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Mathematical Reviews

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NUMBER THEORY

Thébault, Victor. Sur un nouveau théorème d'arithmétique. C. R. Acad. Sci. Paris 213, 967–970 (1941). [MF 9661]

A typical result of this paper is the following. Let $N=123\cdots n$ be a number written in the base $n+1$ with all the digits 1, 2, $\cdots$, n (no zero) placed in the order of increasing magnitude. Let $\lambda=\alpha\beta$ be a two digit number chosen from 0, 1, $\cdots$, n, where $\alpha+\beta$ is a number less than and prime to n. Then the product $N\lambda$ will involve a permutation of the digits 0, 1, $\cdots$, n, each digit occurring once and only once. This and other theorems of this paper were suggested by a question proposed by Laisant [Intermédiare des Mathématiciens 1894, 236]. *I. A. Barnett.*

Segre, B. On a parametric solution of the equation $x^3+y^3+az^3=b$, and on ternary forms representing every rational number. J. London Math. Soc. 18, 31–34 (1943). [MF 9208]

It is shown that the equation $x^3+y^3+az^3=b$, a and b rational numbers, $ab\neq 0$, has an infinity of rational solutions which are rational functions of two parameters. This result is applied to the problem of representing a given rational number r by a given ternary cubic form $f(x, y, z)$. Outside of the trivial case in which f is a product of three linear factors, it is found that the cubic surface $f(x, y, z)=r$ always has infinitely many rational solutions at finite distance, except when the cubic curve $f=0$ has no rational points. *O. Zariski* (Baltimore, Md.).

Reichardt, Hans. Einige im Kleinen überall lösbare, im Grossen unlösbare diophantische Gleichungen. J. Reine Angew. Math. 184, 12–18 (1942). [MF 9034]

The author applies his results on Diophantine equations [Math. Ann. 117, 235–276 (1940); these Rev. 2, 34] to the special equation (1) $x^4+17y^4=z^2$. In order to obtain a basis for the solution of (1) in the field of rational numbers we have to determine which of the following equations have solutions in rational integers: (2) $x^4-68y^4=z^2$, $68x^4-y^4=z^2$, $x^4-17y^4=2z^2$, $17x^4-y^4=2z^2$. While it is trivial that (1) and the first equation (2) have solutions, it seems to be difficult to answer this question for the other three equations. Therefore the author's earlier method is not directly applicable. However, he succeeds by considering the more general problem of finding a basis for the solutions of (1) in the field $\mathfrak{P}(\sqrt{-2})$. It follows from this result that (1) and the first equation (2) have only trivial integral rational solutions while the other three equations (2) have no solutions in rational integers. *A. Brauer.*

Niedermeier, Franz. Ein elementarer Beitrag zur Fermatschen Vermutung. J. Reine Angew. Math. 185, 111–112 (1943). [MF 9593]

Proof that $x^{2\lambda}+y^{2\lambda}=z^{2\lambda}$ is impossible for λ an odd prime such that $\lambda\not\equiv 1 \pmod 8$, x, y, z odd and not divisible by λ. In an addendum the author notes that this result was proved by Kummer [1837] by elementary methods.

L. Carlitz (Durham, N. C.).

Bell, E. T. A type of universal arithmetical forms. Proc. Nat. Acad. Sci. U. S. A. 29, 375–378 (1943). [MF 9766]

Let $\nu_1, \nu_2, \cdots, \nu_k$ be a set of k positive integers all different and let $I^{-1}[\nu_1, \nu_2, \cdots, \nu_k]$ be the set of all nonzero integers which may be represented in the form $x_1^{\nu_1}x_2^{\nu_2}\cdots x_k^{\nu_k}$, where the x_i are integers. A polynomial form $F(u_1, u_2, \cdots, u_n)$ is said to be universal for the set $[\nu_1, \nu_2, \cdots, \nu_k]$ if all the numbers in $I^{-1}[\nu_1, \nu_2, \cdots, \nu_k]$ may be represented by F with no u_i equal to zero. The theorem of this paper states that a form F of such a type may be written

$$\sum_{i=1}^{s} d_i u_1^{\alpha_i}u_2^{\beta_i}\cdots u_n^{\gamma_i}, \qquad d_i\neq 0,$$

$\alpha_i, \beta_i, \cdots, \gamma_i$ integers not less than zero, where at least one of the α_i is zero, at least one of the β_i is zero, $\cdots$, at least one of the γ_i is zero. Also considered in this paper is the number of representations of an integer n by such forms. Some specific instances of the theorem are given for forms containing three terms. *I. A. Barnett* (Cincinnati, Ohio).

Jones, Burton W. Related genera of quadratic forms. Duke Math. J. 9, 723–756 (1942). [MF 7925]

Employing the case $n=1$ of the Siegel formula for the weighted mean $A_0(\mathfrak{S}, \mathfrak{T})$ of the number of representations of a quadratic form $\mathfrak{T}$ in n variables by the genus of a quadratic form $\mathfrak{S}$ in m variables [Ann. of Math. (2) 36, 527–606 (1935)], the author derives [§4] recurrence relations [(11) and (12), pp. 743–744] between the weighted mean $A_0(\mathfrak{S}, pt)$ of the number of representations of pt by the genus of $\mathfrak{S}$ and such weighted means of the numbers of representations of the divisor t of pt by each of the genera "related" [§3] to $\mathfrak{S}$. Sections 6–8 of this paper contain some interesting applications to forms in two, three and four variables. There, among other things, the author obtains for ternary forms $\mathfrak{S}$ a simple expression for $A_0(\mathfrak{S}, t)$ in terms of class number and illustrates the use of his above-mentioned recurrence formulae through the study of numerical examples, obtaining in passing a number of fundamental classical results on the number of representations of integers by certain ternary and quaternary forms. In applications the often troublesome question of the number of classes in a genus can, in some cases, be settled conclusively by means of the useful theorems 7 and 8.

A. E. Ross (St. Louis, Mo.).

Ko, Chao and Wang, S. C. Table of primitive positive quaternary quadratic forms with determinants $\leqq 25$. Acad. Sinica Science Record 1, 54–58 (1942). [MF 8835]

The authors of this paper appear to be unaware of a similar table by S. B. Townes [Ann. of Math. (2) 41, 57–58 (1940); these Rev. 1, 134]. The present table differs from that of Townes by the choice of the cross-product coefficients in the representative (reduced) forms. Page 56 contains the following errata: $D=20$ has (1, 1, 2, 20), should have (1, 1, 2, 10); $D=21$ has (1, 2, 3, 3, 0_4, -1, 0), should have (1, 2, 3, 4, 0_4, -1, 0); in $D=24$, (2, 2, 8, 3, -1,

$0_3, -1, 0)$ should read $(2, 2, 3, 3, -1, 0_3, -1, 0)$. On page 58, the fourth determinant is 9 and not 8. In addition to these errata, we note that the entries $(2, 2, 3, 4, -1_2, 0_3, -1)$ for $D=16$ and $(2, 2, 3, 3, 0, -1_2, 0, -1_2)$ and $(2, 2, 3, 5, -1_2, 0_3, -1)$ for $D=24$ are incorrect, being actually of determinants 28, 11 and 36, respectively. A representative for one of the twenty-four classes of determinant 24 is missing, and the entry $(2, 3, 3, 3, -1, 0_2, -1, 0_2)$ of $D=25$, should read $(2, 2, 3, 3, -1, 0_2, -1, 0_2)$. To restore completeness of the table one may use the following reduced forms from Townes in place of the entries referred to above: the form $(2, 2, 3, 3, 1_2, 0, 1_2)$ for $D=16$, and the forms $(2, 2, 3, 4, 1_2, 0, 1_2)$, $(2, 3, 3, 3, -1_2, 1, 0, 1_2)$, and $(1, 2, 4, 4, 0_2, 2)$ for $D=24$. The count of the number of classes yields 14 for $D=16$ and 24 for $D=24$. We have, in the above, used the well-known notation $(a, b, c, d, r, s, t, u, v, w)=ax_1^2+bx_2^2+cx_3^2+dx_4^2+2rx_2x_3+2sx_1x_3+2tx_1x_2+2ux_1x_4+2vx_2x_4+2wx_3x_4$. A run of K zeros is indicated by 0_K for brevity. One may note that the entry $(1, 1, 4, 4, 0_3, 2)$ for $D=16$ in Townes' table should read $(1, 1, 4, 5, 0_3, 2)$. *A. E. Ross.*

Niven, Ivan. An unsolved case of the Waring problem. Amer. J. Math. **66**, 137–143 (1944). [MF 9945]

The case in which $r=2^n-q-2$, where $3^n=2^nq+r$, $0<r<2^n$, in the determination of $g(n)$ was left untreated by Dickson [Amer. J. Math. **58**, 521–529 (1936)]. This case is considered here and it is shown that $g(n)=I=2^n+q-2$. This completes the determination of $g(n)$ for $n>6$. *H. S. Zuckerman* (Seattle, Wash.).

Rubugunday, R. K. On $g(k)$ in Waring's problem. J. Indian Math. Soc. (N.S.) **6**, 192–198 (1942). [MF 8379]

Dickson's [Amer. J. Math. **58**, 521–529, 530–535 (1936)] and Pillai's [J. Indian Math. Soc. (2) **2**, 16–44 (1936)] solutions of the Waring problem are complete for $k>6$, provided (i) $(3/2)^k-[(3/2)^k]\neq 1-2^{-k}[(3/2)^k]$ and (ii) $(3/2)^k-[(3/2)^k]\neq 1-2^{-k}\{[(3/2)^k]+2\}$. Rubugunday proves by elementary arguments that (i) holds for all k, and that (ii) is equivalent to $2^k-1\nmid 3^k+1$ if $k\geqq 3$. Further, (ii) holds in the cases (a) k even, (b) $9\mid k$, (c) 2^k-1 a prime not equal to 7, (d) 2^k-1 has a prime factor not identical to 1 or 7 mod 24, (e) k and $2k+1$ primes, $k\equiv 3 \bmod 4$, $k>3$, (f) k a multiple of values satisfying (e). The argument depends mainly on the theorem that, for positive odd k, a^k-1 cannot divide b^k-1 unless b is a quadratic residue of all prime factors of a^k-1. Finally it is noted that these cases include all $k\leqq n$, excepting at most s values, where $s\leqq O(1)+h\log n$, $1/h=\log\log 4-\log\log 3$. *G. Pall.*

Linnik, U. V. On the representation of large numbers as sums of seven cubes. Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 218–224 (1943). (English. Russian summary) [MF 9584]

In this paper it is shown that every sufficiently large number may be represented as the sum of seven non-negative cubes. This is an improvement on Landau's result that eight cubes suffice. Landau's proof used results on the representation of integers as the sum of three squares. The present proof uses a theorem on the representation of integers by ternary quadratic forms

$$a^rx^2+b^sy^2+c^tz^2,$$

where r s, and t are odd and a, b, c are primes such that $a\equiv b\equiv c\equiv 1 \pmod 5$, $(ab\mid c)=(-1\mid c)$, $(ac\mid b)=(-1\mid b)$ and $(bc\mid a)=(-1\mid a)$. This theorem is quoted from a previous paper [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 363–402 (1940); these Rev. **2**, 348], which according to the reviewer contained an error. In a footnote in the present paper it is suggested that the reviewer's objections have been met in a paper to appear in C. R. (Doklady) Acad. Sci. URSS. *R. D. James.*

Linnik, U. V. On the "analogy property" of Dirichlet's L-series and Siegel's theorem on $h(\sqrt{-D})$. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 107–109 (1943). [MF 8685]

An outline is given of a proof of Siegel's theorem on $h(-D)$, using a rather complicated analytical apparatus and based on a property of L-series analogous to the fact that, if $\zeta(2s)/\zeta(s)=\sum\lambda(n)n^{-s}$, then $(-D\mid n)=\lambda(n)$ provided no prime factor of n is represented by binary quadratic forms of discriminant $-D$. *G. Pall* (Montreal, Que.).

Linnik, U. V. On a conditional theorem by J. E. Littlewood. C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 122–124 (1942). [MF 8655]

Let $L(s, -\Delta)=\sum(-\Delta\mid n)n^{-s}$, where $-\Delta$ is a fundamental negative discriminant. J. E. Littlewood [Proc. London Math. Soc. (2) **27**, 358–372 (1927)] has proved, on the condition that $L(s, -\Delta)$ has no zeros with real part greater than $\frac{1}{2}$, that there exist infinitely many Δ, such that $L(1, -\Delta)<c_1/\ln\ln\Delta$, where c_1 is a constant. It is now proved without this condition that such a sequence exists satisfying $L(1, -\Delta)<c_2/(\ln\ln\Delta)^{\frac{1}{2}}$. The proof starts with a demonstration that, if P_k is a product of k consecutive primes q (>17) satisfying $(5\mid q)=-1$ and if $N_k=aP_k$, where a $(=7, 11, 13$ or $17)$ is chosen so that $N_k\equiv 3 \pmod 8$, then the number $r(N_k)$ of representations of N_k in $x^2+y^2+z^2+5t^2$ does not exceed $c_3N_k/(\ln\ln N_k)^{\frac{1}{2}}$. By counting the number of solutions t of $80t^2\equiv N \pmod{m^2}$ in the interval $I=[N_k^{\frac{1}{2}}/20, N_k^{\frac{1}{2}}/10]$, it is seen that there are at least $s(>c_4N^{\frac{1}{2}})$ numbers $M_{1k}, \cdots, M_{sk}$ of the form N_k-80t^2 with t in I, which are not divisible by any square r^2 with $r>10^6$, if $N_k>10^{10}$. Then, if $r_3(M)$ is the number of representations of M as a sum of three squares,

$$r(N_k)\geqq c_5\sum_{j=1}^{s} r_3(M_{jk})\geqq c_6\sum_{j=1}^{s} h(-M_{jk}),$$

while, as shown by Dirichlet, if $-\Delta_j$ is squarefree, $h(-\Delta_j)=(\sqrt{\Delta_j}/\pi)L(1, -\Delta_j)$. The result follows. *G. Pall* (Montreal, Que.).

Linnik, U. V. New estimations of Weyl's sums by I. M. Vinogradow's method. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 531–533 (1941). [MF 9607]

This note contains a theorem which is an improvement on the results of Vinogradow concerning exponential sums. It reads as follows. Let $F(x)=Ax^{n+1}+A_0x^n+\cdots+A_{n+1}$, where the A_i are real numbers and $A=a/q+\theta/q\tau$, $(a, q)=1$, $\mid\theta\mid\leqq 1$, $q\leqq\tau$; $S=\sum_{x=1}^{P}e^{2\pi iF(x)}$. Then for $n>2$ and every $\epsilon>0$ we have

$$S=O\{P[P^{-1}p^{t+\epsilon}(1+p^{-n}q\log q)]^{2k/2}+p\},$$

where $p>0$ is an auxiliary variable,

$$t=\{1-[n^{1/3}]/n\}^k\{n(n+1)/2\},$$

and k is an arbitrary positive integer. There is a brief account of the method of proof but few details are given. From this theorem it follows that the Hardy-Littlewood asymptotic formula for the number of solutions of $N=x_1^n+\cdots+x_s^n$ holds when $s>c_1n^{14/3}\log n$, where c_1 is a constant. *R. D. James* (Vancouver, B. C.).

Vinogradow, I. M. An improvement of the estimation of sums with primes. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 7, 17–34 (1943). (Russian. English summary) [MF 8526]

By improving his earlier methods the author obtains more exact estimates of the sums

$$S=\sum_{p\leqq N}\chi(p+k),\quad S'=\sum_{p\leqq N}\chi(p(p+k)),$$

where χ is a nonprincipal character mod q, q is an odd prime, $(k, q)=1$ and $N>1$, namely, $S\ll N^{1+\epsilon}G$, $S'\ll N^{1+\epsilon}G$, $G=(1/q+q/N)^{\frac{1}{2}}+N^{-1/8}$. He also obtains a modification of the case $K=1$ of the result in the following review.

G. Pall (Montreal, Que.).

Vinogradoff, I. M. Improvement of some theorems in the theory of primes. C. R. (Doklady) Acad. Sci. URSS (N.S.) 37, 115–117 (1942). [MF 8653]

The improvement over his 1937–38 methods is illustrated by proving that

$$S_0=\sum_{0<k\leqq K}\sum_{p\leqq N}e^{2\pi i\alpha kp}\ll KNH,$$

where K is an integer, $\alpha=a/q+\theta/q^2$, $(a, q)=1$, $|\theta|\leqq 1$ and $H=N^{\epsilon}((1/q+q/N)^{\frac{1}{2}}+N^{-1/5})$. As an application, if $0\leqq\sigma_1<\sigma_2<1$, the number T of primes $p\leqq N$ such that $\sigma_1\leqq\{\alpha p\}<\sigma_2$ satisfies

$$T-(\sigma_1-\sigma_2)\pi(N)\ll NH.$$

This remains true if p ranges over certain products of primes.

G. Pall (Montreal, Que.).

Lototsky, A. V. Sur l'irrationnalité d'un produit infini. Rec. Math. [Mat. Sbornik] N.S. 12(54), 262–271 (1943). (French. Russian summary) [MF 9587]

A method developed by A. Gelfond [Rec. Math. [Mat. Sbornik] 40, 42–47 (1933)] is used by the author to prove that the entire function $\varphi(z)=\prod_1^{\infty}[1+za^{-n}]$, $a>1$ positive integer, cannot take on a complex rational value for a complex rational value of z distinct from zero and $-a^n$. Use is made of the classical power series

$$\varphi(z)=1+\sum_1^{\infty}\{\prod_1^{n}(a^k-1)\}^{-1}z^n,$$

but the main tool is interpolation at the points $\cdots, 0, a^{k-1}\alpha, -a^k, \cdots$, where $k=1, 2, 3, \cdots$ and α is a given complex rational number. It is shown that the corresponding interpolation series converges to $\varphi(z)$ everywhere. Estimates of the coefficients of this series and a discussion of their arithmetical nature shows that the coefficients must be zero from a certain point on if $\varphi(\alpha)$ is assumed complex rational and this leads to the contradiction that $\varphi(z)$ is a polynomial. There are many disturbing misprints, in particular, in all remainder terms the symbol "O" should be replaced by "o."

E. Hille (New Haven, Conn.).

Hua, Loo-Keng. On Diophantine approximation. C. R. (Doklady) Acad. Sci. URSS (N.S.) 32, 395–396 (1941). [MF 9602]

A brief announcement of results to be published later.

H. S. Zuckerman (Seattle, Wash.).

THEORY OF GROUPS

Levi, F. W. On a theorem of group-theory connected with a problem on paper-folding and with some other problems solved and unsolved. Math. Student 10, 162–165 (1942). [MF 9981]

Tuan, Hsio-Fu. On groups whose orders contain a prime number to the first power. Ann. of Math. (2) 45, 110–140 (1944). [MF 9837]

In continuation and extension of recent work of R. Brauer [see, in particular, Amer. J. Math. 64, 401–440 (1942); these Rev. 4, 1, 2] the author obtains the following results. Suppose that G is a group of order pg for p a prime number prime to g, that there exists a 1:1 representation Z of G whose degree z is smaller than $(2p+1)/3$ and that the p-Sylow subgroups of G are not normal. Then the subgroup V of all the elements of order prime to p in G which permute with the elements in one p-Sylow subgroup is a normal subgroup of G. If G/V is not isomorphic to the linear group $LF(2, p)$, then either $p=3$ and G/V is isomorphic to the symmetric group on 4 or the alternating group on 5 letters, or $p=5$ and G/V is isomorphic to the alternating group on 6 letters, or $p=7$ and G/V is isomorphic to the alternating group on 7 letters. In general V is different from the center of G; but this is the case if Z is irreducible or $z\leqq\frac{1}{2}(p+1)$.

R. Baer (Urbana, Ill.).

Sagastume Berra, Alberto E. The automorphisms of the monomial group of order 6. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. Serie 2: Revista 2, 207–214 (1942). (Spanish) [MF 8373]

In his "Theory of monomial groups" [Trans. Amer. Math. Soc. 51, 15–64 (1942); these Rev. 3, 197] Ore says [p. 16]: "The most difficult problem in this theory is the determination of all automorphisms of the symmetry. . . . These cases have also been completely investigated except in the case $m=6$. Here all the necessary preparations for the calculations have been made but, since the actual solution depends on a somewhat laborious method, the work has not been carried through." This the author does, and shows that theorem 4, page 51, of Ore's paper holds also for $m=6$.

H. S. Wall (Evanston, Ill.).

Tchounikhin, Irène et Tchounikhin, S. A. Sur les groupes p-décomposables. C. R. (Doklady) Acad. Sci. URSS (N.S.) 39, 43–45 (1943). [MF 9717]

A group is said to be decomposable with regard to a prime p, which divides its order, if it is the direct product of its p-Sylow subgroup and another subgroup. The present paper enunciates nine theorems, which are to be proved elsewhere; the following two are typical. (9) A group of order $p^{\lambda}n$, where $(p, n)=1$, is decomposable with regard to p if and only if it contains two composition series, such that the first λ indices of the one and the last λ indices of the other are all equal to p. (4) If a group is not decomposable with regard to any of the k different primes which divide its order, it generally contains k nonisomorphic proper subgroups, whose orders are divisible by the k respective primes, but which are not decomposable with regard to those primes. This fails, however, for three special types of group with $k=2$ or 3.

H. S. M. Coxeter.

Miller, G. A. Subgroups transformed according to a group of prime order. Proc. Nat. Acad. Sci. U. S. A. 29, 311–314 (1943). [MF 9444]

A necessary and sufficient condition that all the noninvariant subgroups of a given group G are transformed under G according to a group of prime order is that the character subgroup of G is of prime index under G. After

considering the possibilities which are associated with the oddness or evenness of this prime index, the author remarks that the smallest degree for which there exists a permutation group having this property is 8, there being one such group of degree 8 and one of degree 9. This may explain why the character subgroup did not receive a special name in the early development of group theory. *G. de B. Robinson.*

Miller, G. A. Special invariant subgroups. Proc. Nat. Acad. Sci. U. S. A. **29**, 308–311 (1943). [MF 9443]

The author recalls the historical use of the terms invariant and characteristic subgroups, commutator subgroup, central and norm or character subgroup in the literature. The author would like to correct the italicized statement appearing on page 310 to read: When a group contains an invariant proper subgroup but no characteristic proper subgroup it is the direct product of simply isomorphic simple groups. *G. de B. Robinson* (Ottawa, Ont.).

Littlewood, D. E. On the number of terms in a simple algebraic form under the symplectic group. Proc. Cambridge Philos. Soc. **39**, 197–199 (1943). [MF 9201]

A formula for the degrees of the irreducible representations of the symplectic group is given. This formula had already been obtained by H. Weyl [Math. Z. **24**, 328–376 (1925), in particular, theorem 4 on p. 342]. *R. Brauer.*

Kontorowitch, P. Sur les groupes à base de partition. Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 56–70 (1943). (Russian. French summary) [MF 8792]

The paper is concerned with groups which can be decomposed in set-theoretical sense into mutually disjoint subgroups. Several simple properties of decompositions of this kind are established in §1. In particular, the author shows that, if the number of components is finite, the group must be finite too. In §2 the author derives necessary and sufficient conditions for decomposability of a group whose center is different from unity. Furthermore, he discusses the structure of groups which are decomposable in the previously defined sense and at the same time can be represented as a direct product of two groups. Finally, §3 is concerned with properties of normalizers of irreducible components of a decomposable group and with conditions under which a decomposable group is of Frobenius type. *W. Hurewicz.*

Whitney, Hassler. Topics in the theory of Abelian groups. I. Divisibility of homomorphisms. Bull. Amer. Math. Soc. **50**, 129–134 (1944). [MF 9898]

Let G and Z be Abelian groups and let h be a homomorphism of G into Z. Necessary and sufficient conditions that h is divisible by an integer m are as follows: (a) that $h(G)$ be divisible by m and (b) that $h(g)=0$ for all elements g in G such that $mg=0$. The author also considers the case where G and Z are topological and h is continuous. The preceding theorem holds in this case provided that G has the property that for each positive integer m the mapping mg of G into itself is interior. The group Z is said to be completely divisible if $mZ=Z$ for any integer m different from zero. The pairing of G and $H=\mathrm{Hom}(G, Z)$ is considered and if Z is completely divisible it is shown that G resolves H completely in a sense defined by the author. The note concludes by applying the results to prove a generalized duality theorem in topology. *D. Montgomery.*

Dubreil, Paul. Sur les problèmes d'immersion et la théorie des modules. C. R. Acad. Sci. Paris **216**, 625–627 (1943). [MF 9988]

The first question considered is the imbedding of a semigroup in a group. It has been demonstrated by Malcev [Math. Ann. **113**, 686–691 (1937)] that this is not always possible. Malcev has also given [Rec. Math. [Mat. Sbornik] N.S. **6(48)**, 331–336 (1939); these Rev. **2**, 7] conditions necessary and sufficient that a semigroup be imbeddable in a group. Dubreil recalls the concept of "regular on the right" due to Ore [Ann. of Math. (2) **32**, 463–477 (1931)]. This condition states essentially that, given a and b, there exists a common multiple $ar=bs$. The author states the theorem: "a semigroup S can be imbedded in a group, of which each element has at least one representation of the form ab^{-1}, a and b in S, if and only if S is regular on the right." He gives three further examples of the influence of this condition on semigroups. In a second section he examines modules over rings which are regular on the right. *H. H. Campaigne* (Washington, D. C.).

Dubreil, Paul. Remarques sur les théorèmes d'isomorphisme. C. R. Acad. Sci. Paris **215**, 239–241 (1942). [MF 9491]

The author presents two theorems of isomorphism for groupoids [as defined by Hausmann and Ore [Amer. J. Math. **59**, 983–1004 (1937)]]. A similar presentation by different methods has been made by O. Boruvka [Math. Ann. **118**, 41–64 (1941); these Rev. **3**, 200]. This note is not complete in itself, but depends for its definitions and proofs on two others [P. Dubreil and L. Dubreil-Jacotin, J. Math. Pures Appl. (9) **18**, 63–95 (1939), in particular, p. 85; P. Dubreil, Contribution à la théorie des demi-groupes, Mém. Acad. Sci. **63**, 3 (1937)].

H. H. Campaigne (Washington, D. C.).

ANALYSIS

Differential Equations

Owens, O. G. A boundary-value problem for a non-linear ordinary differential equation of the second order. Duke Math. J. **10**, 721–731 (1943). [MF 9755]

The paper is concerned with the solution of the boundary problem

$$d^2y/dx^2-\lambda f(x, y)y=g(x, y),$$

$y(0)=0$, $\lim_{x\to\infty} y(x)=0$, for $0\leqq x<\infty$. It proves the existence of such a solution under the hypotheses that λ is sufficiently large and that (i) $f(x, y)$ and $g(x, y)$ are continuous, and $f(x, y)$ and $1/f(x, y)$ are positive and uniformly bounded for $0\leqq x<\infty$, $|y|\leqq Y$; (ii) $f(x, y)$ and $g(x, y)$ satisfy uniform Lipschitz conditions as to y; (iii) finally $\lim_{x\to\infty} g(x, 0)=0$, $\int_0^\infty |g(x, 0)|\,dx<\infty$, and, for some constant a^2, $\int_0^\infty |f(x, 0)-a^2|\,dx<\infty$. The medium of proof is the scheme of successive approximation defined by the equations

$$d^2y/dx^2-\lambda f(x, y_{n-1})y_n=g(x, y_{n-1}).$$

The requisite theorems relative to the linear boundary problem

$$d^2y/dx^2-\lambda f(x)y=g(x),$$

$y(0)=y_0$, $\lim_{x\to\infty} y(x)=0$, are derived in the preliminary considerations. *R. E. Langer* (Madison, Wis.).

Bellman, Richard. The stability of solutions of linear differential equations. Duke Math. J. **10**, 643–647 (1943). [MF 9746]

The author generalizes existing results on the boundedness of solutions of linear differential equations considerably while at the same time giving a much shorter and neater proof. He proves that, if

$$(*) \qquad y^{(n)}(x)+P_1(x)y^{(n-1)}(x)+\cdots+P_n(x)y=0$$

has only solutions which with their first $(n-1)$ derivatives are uniformly bounded for $x\geqq 0$, and if

$$\int_0^\infty |P_k(x)-Q_k(x)|\,dx<\infty,$$

then, if all the $P_k(x)$ in (*) are replaced by the $Q_k(x)$, the new equation also has its solutions and their $(n-1)$ derivatives uniformly bounded for $x\geqq 0$. In particular, if the $P_k(x)$ are all constants, then it is not necessary that $P_1(x)\equiv 0$ for the above result to be true. This means that Bellman's results include a theorem of Cesari which treated only this latter case of constant coefficients.

N. Levinson (Cambridge, Mass.).

Strandhagen, A. G. Use of sine transform for non-simply supported beams. Quart. Appl. Math. **1**, 346–348 (1944). [MF 9912]

This note is concerned with the derivation of the solution of the ordinary linear differential equation

$$(*) \qquad \frac{d^4y}{dx^4}+k^2\frac{d^2y}{dx^2}=q(x)$$

under the boundary conditions $y(0)=y(l)=y''(l)=y'(0)=0$; $q(x)$ is a prescribed function of x in $0<x<l$ such that it is defined as zero in $0<x<b$ and $c<x<l$ and is otherwise a given function of x in $b<x<c$. This derivation is accomplished with the aid of the finite Fourier sine transform. The equation (*) arises in the theory of beams.

A. E. Heins (Cambridge, Mass.).

da Silva Dias, Candido. Application of the theory of analytic functionals to the study of a solution of a differential equation of infinite order. Anais Acad. Brasil. Ci. **15**, 243–253 (1943). (Portuguese) [MF 9734]

The differential equation considered is

$$\sum_{n=0}^{\infty} a_n y^{(n)}(x)=F(x),$$

where the generating function $L(t)=\sum a_n t^n$ is of finite exponential type. The solution obtained is the "semi-local" solution of I. M. Sheffer [Duke Math. J. **3**, 593–609 (1937)]. Since the transformation $L[y(t);x]=\sum a_n y^{(n)}(x)$ is, in L. Fantappiè's terminology, a linear analytic functional of mixed type, the author is able to some extent to apply Fantappiè's theory of such functionals, but in general the reasoning parallels that in Sheffer's paper.

J. V. Wehausen (Columbia, Mo.).

de Toledo Piza, Affonso P. Determination of solutions of Fourier's partial differential equation by a method called refinement of solutions. Anais Acad. Brasil. Ci. **15**, 325–342 (1943). (Portuguese) [MF 10047]

The author gives an integral transformation which changes a known solution of the equation $z_{xx}+x^{-1}z_x=z_y$ into another solution of the same equation. An infinite sequence of solutions is obtained in this way in a very elegant manner by starting from the obvious one $z=0$ and applying the method successively. The general formulae are too lengthy to be given here. *I. Opatowski* (Chicago, Ill.).

Minakshisundaram, S. On expansion in eigenfunctions of boundary value problems. V. J. Indian Math. Soc. (N.S.) **7**, 89–95 (1943). [MF 10105]

[Parts I–III appeared in Proc. Indian Acad. Sci., Sect. A. **12**, 462–465 (1940); J. Indian Math. Soc. (N.S.) **5**, 103–108 (1941); **6**, 153–167 (1942); these Rev. **3**, 128, 244; **4**, 246. Part IV has not yet been published. For part VI see the following review.]

Let D be a finite domain of the x, y plane bounded by a curve C consisting of a finite number of regular arcs. Let $\mu_1, \mu_2, \mu_3, \cdots$ be the eigenvalues of the boundary value problem

$$\nabla^2\omega+\mu\omega=0 \text{ in } D, \qquad \omega(x,y)=0 \text{ on } c,$$

and let $\omega_1(x,y), \omega_2(x,y), \omega_3(x,y), \cdots$ be the corresponding eigenfunctions. We set $\lambda_n=\mu_n^{\frac{1}{2}}$, $n=1, 2, 3, \cdots$. This paper is concerned with topics connected with the absolute convergence of the Fourier series of a function $f(x,y)$, that is, the series $\sum a_n\omega_n(x,y)$, where $a_n=\iint_D f(x,y)\omega_n(x,y)\,dxdy$. A restriction is imposed on the behavior of $f(x,y)$ near C, and it is also assumed that $f(x,y)$ is summable on any continuous curve lying in D, in particular on circles lying in D. By functions $f(x,y)\in \mathrm{Lip}(\alpha, 2)$ in D we mean those satisfying the condition

$$\iint_D \{f(x+h,y+k)-f(x,y)\}^2\,dxdy=O\{(h^2+k^2)^\alpha\}, \quad h,k\to 0.$$

The main results given in the paper are contained in the following two theorems: If $f(x,y)\in \mathrm{Lip}(\alpha, 2)$ in D, then $\sum|a_n\omega_n|^\beta$ is convergent for $\beta>4/(2\alpha+1)$ and $\frac{1}{2}<\alpha\leqq 1$. If $f(x,y)$ is piecewise continuous in D and if $\partial f/\partial x$ and $\partial f/\partial y$ exist and belong to Lip $(\alpha, 2)$, then we have

(i) $a_n=O(\lambda_n^{-\alpha-1})$;
(ii) $\sum \mu_n^{\alpha+1-\epsilon}a_n^2$ is convergent ($\epsilon>0$ and arbitrary);
(iii) $\sum|a_n\omega_n|^\beta$ is convergent for $\beta>4/(2\alpha+3)$;
(iv) $\sum \mu_n^\gamma|a_n\omega_n|$ is convergent for $\gamma<\frac{1}{2}\alpha-\frac{1}{4}$.

F. W. Perkins (Hanover, N. H.).

Minakshisundaram, S. On expansion in eigenfunctions of boundary value problems. VI. Bull. Calcutta Math. Soc. **35**, 81–84 (1943). [MF 9954]

[Cf. the preceding review.] The author outlines generalizations of many results obtained in his earlier papers, making them applicable to the expansion of arbitrary functions of m variables, $x_1, x_2, \cdots, x_m$, in series of eigenfunctions associated with the boundary value problem

$$\sum_{i=1}^{m}\frac{\partial^2 w}{\partial x_i^2}+\mu w=0 \text{ in } D, \quad w(x_1, x_2, \cdots, x_m)=0 \text{ on } C,$$

where C is the boundary of a suitable m-dimensional domain D.

F. W. Perkins (Hanover, N. H.).

de Groot, S. R. Théorie phenomenologique de l'effet Soret. Physica **9**, 699–708 (1942). [MF 9824]

Let $n(x,t)$ denote the concentration of a substance in a solution between two parallel plates, where x is the distance from one of the plates and t is the time. If the initial concentration of the given substance is uniform a temperature gradient in the solution, produced by a difference in temperatures of the plates, is accompanied by a change in $n(x,t)$. Assuming a constant temperature gradient, and

assuming that the flux of the substance is a linear function of n and $\partial n/\partial x$, the author shows that n satisfies a form of the heat equation with constant coefficients. The boundary conditions at the plates involve both n and $\partial n/\partial x$. He solves the boundary value problem by classical methods and obtains a formula for $n(x, t)$. Some discussion is given on the difficulty of applying the solution of this problem to known observed results. *R. V. Churchill.*

Vasilesco, Florin. Sur quelques critères nouveaux de stabilité. C. R. Acad. Sci. Paris **215**, 296–297 (1942). [MF 9501]

The author states, without proofs, extensions, with more general mass distributions, of theorems given by de la Vallée Poussin [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **24**, 368–384, 672–689 (1938)] and M. Brélot [ibid. (5) **25**, 125–137 (1939)]. The following is to illustrate the type of results given. If μ_n is a positive mass distribution with respect to the sets T_n of de la Vallée Poussin [loc. cit.] where their potentials v_n are almost everywhere (à peu près partout) uniformly bounded, its lower bound, especially, being positive, then the point P is regular or irregular according to the divergence or convergence of the series $\sum_1^\infty v_n(P)$. *František Wolf* (Berkeley, Calif.).

Vasilesco, Florin. Sur quelques formes plus générales des critères de régularité de MM. de la Vallée Poussin et Wiener. C. R. Acad. Sci. Paris **215**, 249–251 (1942). [MF 9492]

The author states, without proofs, generalizations of results given by M. Brélot [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **25**, 125–137 (1939); these Rev. **1**, 238] and extensions to points of stability of results similar to those for regular points given by Wiener. *František Wolf.*

Kravtchenko, Julien. Sur la continuité des dérivées du potentiel. C. R. Acad. Sci. Paris **213**, 676–679 (1941). [MF 9650]

The author announces the result of the problem: given the mode of continuity of the mass distribution and of the space, determine the mode of continuity of the derivatives of the potential created by the masses. If the continuity is of Hölder's type, the result is very simple and well known. The author uses an integral involving a singular kernel and the function which specifies the mode of continuity of the space and the mass distribution. The generalized result contains the special case of logarithmic continuity. *H. S. Tsien* (Pasadena, Calif.).

Cartan, Henri. Capacité extérieure et suites convergentes de potentiels. C. R. Acad. Sci. Paris **214**, 944–946 (1942). [MF 9469]

The author is concerned with potentials of positive mass distributions. It has been known that the set of points at which the potential is ∞ without being ∞ identically (ensemble polaire) is of interior capacity zero. The author announces the result that a set has the above property if, and only if, its exterior capacity is zero. Further, if u_n denotes a monotone sequence of potentials, bounded from above by a fixed potential, and if v is a potential equal to $\lim u_n$ except at the points of a set of capacity zero, then, for any $\alpha>0$, the capacity of the set $\phi_n=E[|v-u_n|>\alpha]$ tends to zero as $n\to\infty$. Another result is that the space of positive mass distributions is complete if, as its distance, we introduce the energy of the field generated by the difference of any two distributions. *František Wolf.*

Cartan, Henri. Sur les suites de potentiels de masses ponctuelles. C. R. Acad. Sci. Paris **214**, 994–997 (1942). [MF 9474]

The author continues to give results on the subject treated in the paper reviewed above. The property of a set E, of having its exterior capacity zero, is a necessary and sufficient condition in order that there should exist a sequence of superharmonic functions $u_n\geqq 0$, such that $\liminf u_n$ is ∞ at the points of E without being identically ∞. If the condition is satisfied, then we can take for u_n a convenient potential generated by a finite number of point charges. Further, if we have

$$v(x)=\liminf_{y\to x} u(y), \quad u(x)=\lim u_n(x)$$

and $u_n(x)$ continuous in a large sense at points where $u_n(x)=\infty-$ and superharmonic, then the necessary and sufficient condition that there should exist such $u_n(x)$ that $u(x)\neq v(x)$ at the points of E is that E should be the sum of an enumerable number of compact sets of zero capacity. The $u_n(x)$ can again be potentials of a finite number of point charges. *František Wolf* (Berkeley, Calif.).

Teissier du Cros, F. Sur l'ensemble des fonctions sous-harmoniques et l'ensemble des fonctions harmoniques dans un cercle. C. R. Acad. Sci. Paris **216**, 437–438 (1943). [MF 10023]

The set of all functions continuous and subharmonic in the interior of a circle C have the following properties. (1) If u and v are functions of the set, and if a and b are nonnegative constants, then $au+bv$ is a function of the set. (2) If u is a function of the set and if m is an interior point of a circle completely interior to C, then any conformal transformation which carries c into C transforms $u(m)$ into $U(M)$, a function of the set. (3) In c, each function of the set attains its least upper bound on the circumference. The author shows also that any set of functions having these properties consists exclusively of subharmonic functions. He further notes that, if the requirement that a and b be nonnegative be deleted from (1), then the set of functions consists exclusively of functions harmonic in C. [N.B. In the third line of the section on harmonic functions, for "propriétés 1° et 2°," read "propriétés 2° et 3°."] *F. W. Perkins* (Hanover, N. H.).

Functional Analysis, Ergodic Theory

Kakutani, Shizuo and Mackey, George W. Two characterizations of real Hilbert space. Ann. of Math. (2) **45**, 50–58 (1944). [MF 9834]

Real Hilbert space is characterized in terms of the lattice of closed linear subsets and in terms of the set of all bounded linear operators. Thus, if a complete real Banach space X is such that the lattice of all closed linear sets "admits an involutory anti-isomorphism," that is, an operation analogous to that of taking the orthogonal complement, then X is a Hilbert space. In the set of operators, the operation of taking the adjoint has a similar role. The authors also give a number of examples relative to the assumption of completeness for X. The last of these is obscure to the reviewer. *F. J. Murray* (New York, N. Y.).

Day, Mahlon M. Uniform convexity. III. Bull. Amer. Math. Soc. **49**, 745–750 (1943). [MF 9316]

[Parts I and II appeared in the same Bull. **47**, 313–317, 504–507 (1941); cf. these Rev. **2**, 221, 314.] It is shown

that a Banach space which is locally uniformly convex near some point b (in the sense that the condition for uniform convexity holds in some sphere about b) is isomorphic to a uniformly convex space. Another result gives a necessary condition for isomorphism with a uniformly convex space. For a suitably restricted Banach space T of real functions on a range S the following result is established. Let B_s, $s \in S$, be Banach spaces and $P_T(B_s)$ consist of those functions $b=b_s$ with $b_s \in B_s$ and $(\|b_s\|) \in T$. Define a norm in $P_T(B_s)$ by $\|b\| = \|(\|b_s\|)\|$. The space $P_T(B_s)$ is uniformly convex if and only if T is uniformly convex and the spaces B_s have a common modulus of convexity. *N. Dunford.*

Julia, Gaston. Sur une classe d'opérateurs bilinéaires bornés de l'espace hilbertien. C. R. Acad. Sci. Paris **212**, 1059–1062 (1941). [MF 9635]

The author establishes in this paper the proposition that, if A_n is a sequence of bounded linear operators on a Hilbert space H, if e_n is a complete orthonormal basis for H and if $B(x, y) = \sum (x, e_i) A_i(y)$ converges strongly for each x and y in H, then this sum is a bounded linear operator on H. The author has noticed that this theorem as well as a number of other results such as the theorems of Landau and Hellinger-Toeplitz in Hilbert space are special cases of a generalization of a result of Osgood: if a sequence of lower semi-continuous functions converges on a domain of H, it is uniformly bounded on a sphere in the domain. This generalized Osgood theorem is a special case of one of the reviewer on a theorem of Hildebrandt [Studia Math. **7**, 157–158 (1938)], and the continuity of $B(x, y)$ is also an immediate consequence of Hildebrandt's theorem itself [Bull. Amer. Math. Soc. **29**, 309–315 (1923)]. Another general proposition of the Osgood type was given by Banach [Théorie des opérations linéaires, Warsaw, 1932, p. 19]. *H. H. Goldstine* (Aberdeen, Md.).

Milman, D. Sur une classification des points du spectre d'un opérateur linéaire. C. R. (Doklady) Acad. Sci. URSS (N.S.) **33**, 279–281 (1941). [MF 9570]

Let E be a complex Banach space, A a linear transformation in E, S_A the spectrum of A and I the identity transformation. The author defines a point $\lambda_0 \in S_A$ to be "essentially of minimal multiplicity n" if (1) for $x_m \in E$, $\lim \|(A-\lambda_0 I)^{n+1} x_m\| = 0$ implies $\lim \|(A-\lambda_0 I)^n x_m\| = 0$; (2) there exists $x_m^0 \in E$ such that $\lim \|(A-\lambda_0 I)^n x_m^0\| = 0$ and $\overline{\lim} \|(A-\lambda_0 I)^{n-1} x_m^0\| > 0$. A point $\lambda_0 \in S_A$ is of "minimal multiplicity n" when (1) and (2) hold with the additional restriction that $\|x_m\| = 1$. Of several theorems concerning such points in S_A we quote two. If $\lambda_0 \in S_A$ is essentially of minimal multiplicity n and if λ_0 is a limit point of the complement of S_A, then $\|(A-\lambda I)^{-1}\| \cdot |\lambda_0 - \lambda|^n < K$, where $\lambda \bar{\in} S_A$ lies in a certain neighborhood of λ_0 and K is a constant. For the second theorem it is assumed that for every $\lambda \bar{\in} S_A$ there exists a rectifiable closed contour containing S_A and excluding λ. If $\lambda_0 \in S_A$, $\lambda_m \bar{\in} S_A$, $\lambda_m \to \lambda_0$ and $\lim \|(A-\lambda_m I)^{-1}\| \cdot |\lambda_0 - \lambda_m|^{n+1} = 0$, then λ_0 is of minimal multiplicity not greater than n. Proofs are not given. *J. V. Wehausen.*

Arnold, B. H. Rings of operators on vector spaces. Ann. of Math. (2) **45**, 24–49 (1944). [MF 9833]

Let V denote a vector space, with complex scalar multiplication and having a prescribed set of linear functionals, that is, in the more customary terminology, a topological linear space. Let R denote the ring of bounded operators on V. It is shown that two such spaces V_1 and V_2 are isomorphic if the R_1 and R_2 are isomorphic. This is a generalization of a result of Eidelheit for real Banach (normed) spaces. The essential addition is to establish that the one-to-one isomorphism of the centers of R_1 and R_2 is continuous. Let M denote a ring of operators in Hilbert space H. [Cf. J. von Neumann, Math. Ann. **102**, 370–427 (1929).] Using certain known results on factors in the theory of rings of operators in Hilbert space, the author shows that, if R is isomorphic to such an M, then V and H are isomorphic.

The author also considers topologies for R (and for V) which involve limits based on directed systems of elements $\{\alpha\}$. Four such topologies are given, the weak, the strong, the "finite" and the "finite*." The first two are analogous to the usual notions. A directed system A_α of operators of R is convergent in the finite topology if, for each $x \in V$, there is an α^* such that if $\alpha > \alpha^*$ then $A_\alpha x = A_{\alpha^*} x$. Convergence in the finite* topology means that, for every $f \in V^*$, there exists an α^* such that if $\alpha > \alpha^*$ then $f(A_\alpha x) = f(A_{\alpha^*} x)$ for every $x \in V$. For each topology the completion of R is discussed. A sufficient set of conditions on R is given that V be a conjugate space. These conditions demand that a subset R' of R be complete in the finite* topology and also involve the finite dimensional operators of R'.

There is also a discussion of the reflexiveness of V in terms of the finite* topology. However the reviewer does not understand the final result of this section: "Corollary 2. A necessary and sufficient condition that the strong completion of V be reflexive is that the strong and finite* topologies in R agree." It seems to the reviewer that the following is a counter-example in Hilbert space for the necessity. Let $\{E_n\}$ be a sequence of increasing projections with finite dimensional ranges such that E_n approaches I in the strong topology. However I is not the limit of E_n in the finite* topology. From the proof, the reviewer believes that what is meant is that certain completions agree, rather than the topologies. *F. J. Murray.*

Gelfand, I. and Neumark, M. On the imbedding of normed rings into the ring of operators in Hilbert space. Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 197–213 (1943). (English. Russian summary) MF 9583]

[For Gelfand's theory of normed vector rings see, in particular, Rec. Math. [Mat. Sbornik] N.S. **9(51)**, 3–24 (1941); these Rev. **3**, 51.] In the present paper multiplication is normally noncommutative. The authors consider *-rings in which to every element x there is an adjoint x^* such that (1) $(\alpha x + \beta y)^* = \bar{\alpha} x^* + \bar{\beta} y^*$, (2) $x^{**} = x$, (3) $(xy)^* = y^* x^*$, (4) $\|x^* x\| = \|x\|^2$, (5) $x^* x + e$ has an inverse. An element h is Hermitian if $h^* = h$; every $x \in R$ is of the form $x = h_1 + i h_2$ (unique representation). The spectrum of h is real; if positive, h is said to be positive. If h is positive, $h = g^2$, where g is positive. If h_1 is positive and regular, then $h_1 + i h_2$ is regular for all h_2. The sum of positive elements is positive and so is the product if the first factor is regular.

The main theorem asserts that a *-ring R can always be mapped on a bounded closed subring of the ring B of all linear bounded operators on a Hilbert space to itself, the mapping being an isometric isomorphism which takes adjoints into adjoints. The space $\mathfrak{H}$, which is not necessarily separable, is found by an ingenious construction. In the special case in which R is simple, that is, contains no proper two-sided ideal, the elements of $\mathfrak{H}$ are the residue-classes X of R, modulo a left maximal ideal M, and the inner product $(X, Y) = f(y^* x)$, $x \in X$, $y \in Y$. Here $f(z)$ is a linear functional on R such that $f(e) = 1$, $f(x) \geqq 0$ if x is positive

Hermitian and $f(x)=0$ if $x \varepsilon M$. If the resulting space is not complete under the norm $(X, X)^{\frac{1}{2}}$, it is extended to a complete space $\mathfrak{H}$, which is the required Hilbert space. Let $a \varepsilon R$ and let the elements $ax, x \varepsilon X$, belong to the residue class AX. Then $X \rightarrow AX$ is a bounded linear transformation on $\mathfrak{H}$ to $\mathfrak{H}$ corresponding to $a \varepsilon R$. The correspondence between a and A turns out to be an isomorphism, $\|a\| = \|A\|$, and the set of all transformations $A = A(a)$ in B is the desired image of R. In case R is not simple, the construction leads to a space $\mathfrak{H}_M$ for each M and $\mathfrak{H}$ is taken as the direct sum of all these spaces while A is defined as A_M in $\mathfrak{H}_M$. The correspondence $A = A(a)$ has the required properties.

If the image of R in B is weakly closed, further results come out which connect with the factorization theory of F. J. Murray and J. von Neumann [see, in particular, Ann. of Math. (2) **37**, 116–229 (1936)]. In this theory a self-adjoint subring is a factor if its center is the scalar field. The class of a factor is determined by the range of the relative dimension function $\mathfrak{D}(\mathfrak{M})$ of its closed linear subsets. The present authors show that every simple weakly closed subring of B is a factor. Conversely, they prove that every factor of class I_n $[\mathfrak{D}(\mathfrak{M}) = 0, 1, \cdots, n]$ is simple and, if the space is separable, also every factor of class III $[\mathfrak{D}(\mathfrak{M}) = 0, \infty]$. On the other hand, in a separable Hilbert space, the factors of class I_∞ $[\mathfrak{D}(\mathfrak{M}) = 0, 1, \cdots, n, \cdots]$ and II_∞ $[\mathfrak{D}(\mathfrak{M}) = [0, \infty]]$ are not simple. Such a factor has a single proper two-sided ideal closed in the uniform topology which contains all finite projections and coincides with the set of all compact, that is, completely continuous, operators of the factors. The latter result contains results due to J. W. Calkin [Ann. of Math. (2) **42**, 839–873 (1941); these Rev. **3**, 208]. The authors also point out that their results imply that the operator ring theory developed by S. W. P. Steen in 1936–1940 can be derived from corresponding results of F. J. Murray and J. von Neumann for operators in a Hilbert space. *E. Hille.*

Hurewicz, Witold. Ergodic theorem without invariant measure. Ann. of Math. (2) **45**, 192–206 (1944). [MF 9839]

Let S be a space in which a measure is defined by means of a countably additive nonnegative set function μ defined on a Borel family of (the measurable) subsets. It is assumed that S is a countable sum of sets of finite measure. The 1-1 transformation T of S onto itself is measurable if, X being measurable, the sets $T(X)$ and $T^{-1}(X)$ are measurable. A measurable set X is a nullset with respect to T if $\mu(T^n(X)) = 0$ for all integral n. A measurable set X is a wandering set with respect to T if $XT^n(X) = 0$, $n = 1, 2, \cdots$. The countably additive set function $F(X)$, defined on the measurable subsets of S, is absolutely continuous with respect to the measure μ if $\mu(X) = 0$ implies $F(X) = 0$, and $\mu(X) < \infty$ implies $|F(X)| < \infty$. Let the set functions μ_n and F_n be defined by

$$\mu_n(X) = \sum_{i=0}^{n} (T^i(X)), \quad F_n(X) = \sum_{i=0}^{n} F(T^i(X)), \quad n = 1, 2, \cdots.$$

These set functions are countably additive, μ_n is nonnegative and can therefore be regarded as a measure function and, under the assumption that F is absolutely continuous with respect to μ, F_n is absolutely continuous with respect to the measure μ_n. It follows that there exists a finite function $f_n(x)$ such that $F_n(X) = \int_X f_n(x) d\mu_n$. The sequence of functions $f_1, f_2, \cdots$ thus associated with the set function F is termed the averaging sequence belonging to F.

The main result of this paper is the following ergodic theorem. If S contains no set of positive measure which is wandering with respect to T and $F(X)$ is a finite countably additive set function defined on the measurable sets such that F is absolutely continuous with respect to the measure μ, then the averaging sequence belonging to F converges on S except possibly on a nullset. The limit function $f(x) = \lim_{n \to \infty} f_n(x)$ is defined except on a nullset and satisfies the following conditions: (a) f is invariant except on a nullset; (b) f is summable over S; (c) if X is measurable, $T(X) = X$ and $\mu(X) < \infty$, then $\int_X f(x) d\mu = F(X)$.

It is to be noted that it is not assumed that the transformation T is measure preserving. It is easily shown that the stated theorem together with the hypothesis that T is measure preserving imply the G. D. Birkhoff ergodic theorem and its generalization due to E. Hopf [cf. E. Hopf, Ergodentheorie, Springer, Berlin, 1937, p. 49]. The proof of the stated theorem is straightforward and elegant and the methods used are to some extent similar to the original methods of G. D. Birkhoff. The author promises to treat continuous flows in a subsequent paper. *G. A. Hedlund.*

TOPOLOGY

Sagastume Berra, A. E. Note on p-adic numbers and topology. An. Soc. Ci. Argentina **133**, 218–221 (1942). (Spanish) [MF 8427]

The author states without proofs properties of a topology defined in a field by means of a function $o(x)$ over the field. *H. S. Wall* (Evanston, Ill.).

Appert, Antoine. Groupes ($\mathfrak{B}$) et espaces complètement quasi réguliers. C. R. Acad. Sci. Paris **216**, 679–681 (1943). [MF 9990]

Topological groups are considered in which the topology of the underlying space is given by a neighborhood space of Fréchet. It is shown that the fact that the space is a topological group necessarily imposes certain restrictions on the topology of the space. For example, if a point and a closed set are disjoint, they are interior to disjoint sets. Several other results of this nature are obtained. The author mentions that previous writers have given special cases of his results. [See, for instance, Pontrjagin, Topological Groups, English translation, Princeton University Press, Princeton, N. J., 1939, p. 54; cf. these Rev. **1**, 44.] *D. Montgomery* (Princeton, N. J.).

Ehresmann, Charles. Espaces fibrés associés. C. R. Acad. Sci. Paris **213**, 762–764 (1941). [MF 9654]

This is the missing link between two notes of the same author (one of them jointly with J. Feldbau) which have already been reviewed here [same C. R. **212**, 945–948 (1941); **214**, 144–147 (1942); these Rev. **3**, 58; **4**, 146]. The definition of fibre-spaces is the same as there, that is, substantially the same as H. Whitney's, depending as it does on the use of a given transitive group of homeomorphisms of the fibre (or rather of a space homeomorphic to all the fibres) into itself. The latter group being denoted by G, and the base-space being denoted by B, the author introduces what he calls the principal fibre-space associated

with the given one, which is a fibre-space with the same base-space B, and with fibres which are homeomorphic to the group-space G (considered as homogeneous space under the group of left-hand translations); the original fibre-space can itself be derived from this one by suitable identifications of elements. A criterium is given, in terms of the principal fibre-space, for the original space to be a direct product; in particular, by the deformation lemma of the earlier note, this criterium is always satisfied when the base-space can be contracted into a point. *A. Weil* (Bethlehem, Pa.).

Goffman, Casper. On linear spaces which may be rendered complete normed metric spaces. Bull. Amer. Math. Soc. 49, 611–614 (1943). [MF 8863]

It is shown that a linear space is isomorphic with a linear normed metric space which is complete if and only if the cardinal number of its Hamel basis is not the limit of an enumerable sequence of smaller cardinals. The generalized continuum hypothesis is used in the proof. *L. W. Cohen.*

Ribeiro, Hugo. Sur les espaces à métrique faible. Portugaliae Math. 4, 21–40 (1943). [MF 8630]

A space is said to have a weak metric ρ if $\rho(x, x)=0$ and $0\leqq\rho(x, z)\leqq\rho(x, y)+\rho(y, z)$ whenever x, y and z are elements of the space. The author is interested in the weak metrization problem and has the following theorems, the second of which shows how semi-continuous functions are connected with this problem in much the same way that continuous functions are connected with the metrization problem. Theorem 1. A Kuratowski space satisfying the second axiom of denumerability is weakly metrizable. Theorem 2. A necessary and sufficient condition that a neighborhood space (V) (of Fréchet) be weakly metrizable is that there exists a family of functions defined in the space and equi-semi-continuous in the space and for each point and each neighborhood of the point there is at least one function of the family which is zero at the point and whose infimum on the complement of the neighborhood is positive. Theorem 3. A space (V) is weakly metrizable if and only if there is a family of neighborhoods topologically equivalent to the original family and such that in this family the neighborhoods of a point are denumerable and ordered in such a way that a neighborhood of order m of any point in the neighborhood of order n of any point is contained in the neighborhood of order $\min(m, n)-1$ of the latter point. Theorem 4. A necessary and sufficient condition that a space (V) have a weak metric is that the family of neighborhoods contains a topologically equivalent subfamily of neighborhoods such that the neighborhoods of each point are denumerable and such that for each point a and each integer n there is an integer $m>n$ for which $U_m(b)\subset U_n(a)$ whenever $b\in U_m(a)$. Also the theorem that a compact Hausdorff space having a weak metric is a metric space is proved and a question raised as to whether certain conditions together with normality of the space imply a metric. The quasi-metrics of Wilson and Albert are included in an exposition of the weak metrization problem and its relation to the metrization problem. *J. F. Randolph.*

White, Paul A. r-regular convergence spaces. Amer. J. Math. 66, 69–96 (1944). [MF 9940]

This paper considers the hyperspace $K=K^r$ of lc^r closed subsets of a compact metric space M, where r is an arbitrary integer. The major part of the work is devoted to establishing a metric in K where regular convergence as introduced by G. T. Whyburn [Fund. Math. 25, 408–426 (1935)] is used to define the limit concept in K. The necessity of a metric in K other than the usual Hausdorff metric follows from the fact that under this metric the space K would not be closed in the space H of all closed subsets of M. It is shown that K can be metrized by means of the Hilbert metric in such a way that the regular convergence definition of a limit point is preserved.

Some time is spent in considering what properties M will have if K is specialized and conversely. It is shown, for example, that K is connected if and only if M is a single point, while K has a finite number of components if and only if M is a finite set. In order that two points p and q may be joined by an arc in K it is necessary and sufficient that the lc^r subset P of M be $\leqq r$-regularly deformable into Q (for definition, see the paper). The properties that K be compact, M be finite, and $H=K$ are equivalent. Among a good many other properties established the following serves as an example: If M is a continuum containing no non-0-regular convergent sequence of arcs then (a) M contains no simple closed curve, and if M is arcwise connected it is a dendrite, (b) for each point p the set D_p consisting of all points q of M such that there is an arc pq in M is a dendrite (D_p is the maximal locally connected subset of M containing p), (c) the sets $[D_p]$ form an upper semi-continuous decomposition of M.

S. Mazurkiewicz [Fund. Math. 24, 118–134 (1935)] has considered the hyperspace of locally connected subcontinua of a locally connected compact metric space M, and obtained a metric equivalent to that of the author but phrased differently. The Mazurkiewicz metric is complete whereas the White metric is complete only in very special cases. The White metric, however, has the advantage of being defined in a less restricted space. *D. W. Hall.*

Fan, Ky. Quelques propriétés caractéristiques des ensembles possédant la propriété des quatre points et des ensembles filiformes. C. R. Acad. Sci. Paris 216, 553–555 (1943). [MF 10028]

A set is said to possess the property of four points if it is connected, contains more than one point and if whenever a, b, c, d are four distinct points of the set it is true that one of the three pairs a, b or a, c or b, c separate the respective remaining pair. A set E is said to be thread-like (filiforme) if it is connected, contains more than one point and if whenever three distinct points are given at least one of them separates the other two. This note gives necessary and sufficient conditions for the existence of these properties. Let E be a space of F. Riesz which is connected and locally connected. In order that E possess the property of four points each of the two following conditions is necessary and sufficient. (1) If three domains in E have a point in common then at least one of them is contained in the union of the other two. (2) If three domains are given whose union is a proper subset of the space then at least one of them is contained in the product of the two others. New characterizations are found for fundamental figures, that is, spaces which are homeomorphic to an arc, a line, a ray or a circumference. *D. Montgomery* (Princeton, N. J.).

Moore, R. L. Concerning continua which have dendratomic subsets. Proc. Nat. Acad. Sci. U. S. A. 29, 384–389 (1943). [MF 9768]

A subset K of a compact continuum M is a dendratomic subset provided it is an element of some upper semicon-

tinuous decomposition G of M into continua with a dendron as its hyperspace such that for any upper semicontinuous decomposition H of M with a dendron as hyperspace each element of G is a subset of some element of H. A compact continuum M is a web provided there exist two upper semicontinuous decompositions H_1 and H_2 of M into continua having dendrons as hyperspaces and such that there are uncountably many elements of H_1 no one of which is contained in any single element of H_2. In this paper a study is made of these two notions and among other things it is shown that a compact continuum has dendratomic subsets if and only if it is not a web. Also, every web is a triod, that is, the sum of three continua which intersect by pairs in a fixed continuum. Further, every compact continuum M which is not a triod is shown to have dendratomic subsets and these actually are "arcatomic." In particular, if M is irreducible between two of its points it has such subsets.

G. T. Whyburn (Charlottesville, Va.).

Moore, R. L. Concerning webs in the plane. Proc. Nat. Acad. Sci. U. S. A. **29**, 389–393 (1943). [MF 9769]

Let M denote a compact plane continuum. In this paper it is shown that if M contains a web it must be a web [see the preceding review for definition]. Also, if G and H are upper semicontinuous decompositions of M into continua such that the intersection of any $g \in G$ with any $h \in H$ is a nonempty totally disconnected set, then the hyperspaces of both G and H are continuous curves. Examples are given showing that neither of these conclusions holds for continua in three dimensional space. *G. T. Whyburn.*

Choquet, Gustave. Structure des domaines plans et accessibilité. C. R. Acad. Sci. Paris **216**, 279–280 (1943). [MF 10012]

This note seeks an answer to the question of when two plane domains have the same form. Let Δ be a bounded open plane domain and let Φ be its boundary. Associated with this domain there are the four following topological spaces. (a) The space U is identical with Δ. (b) The space V has for points the points of Δ and the accessible points of Φ. Neighborhoods of a point M in V have for a base sets of points which may be joined to M by arcs of diameter less than a given number. (c) The space W has for points those of Δ. The distance between two points M and M' of W is the lower bound of diameters of arcs joining M and M'. The space W may be completed to a space $\hat{W}$. (d) The space X has for points those of Δ. The distance of two points is defined to be their Euclidean distance. The completed space $\hat{X}$ is homeomorphic to $\Delta+\Phi$. Two domains Δ_1 and Δ_2 are said to have the same form according to U if U_1 and U_2 are homeomorphic (same for V, W, X). The four conditions thus introduced are progressively more restrictive. Conditions for the homeomorphism of the various associated spaces, including $\hat{W}$ and $\hat{X}$, are stated.

If Δ is simply connected a cyclic order may be defined on the accessible points of Φ and the author considers the points in two sets K_1 and K_2 corresponding to two complementary arcs of the cycle of order. The points of Δ which are the centers of circles touching K_1 and K_2 form an arc, called the arc of equidistance of K_1 and K_2 or a curve γ. These curves form a two parameter family analogous to a family of geodesics. The intersection of two such curves is a point or an arc. Two curves γ having an extremity in common on Φ have (except in a trivial case) in common an arc of extremity M. *D. Montgomery* (Princeton, N. J.).

Choquet, Gustave. Topologie de la représentation conforme. C. R. Acad. Sci. Paris **216**, 330–331 (1943). [MF 10015]

Let OA be a simple arc for which every proper subarc OP is rectifiable; let l be the length of the arc OP and let $\delta(l)$ be the diameter of the arc PA. Let $f(x)$ be a continuous increasing function of x which is zero when $x=0$. The value of the integral

$$\int_0^A f[\delta(l)]dl$$

is called the "sinuosity" of the arc OA at the point A for $f(x)$. Let M be an accessible point of Φ [see preceding review for definitions], α an arc of access to M and $f(x)$ a function of the type described. If α has finite sinuosity at M for $f(x)$ then so do the arcs of equidistance of extremity M. Further definitions and theorems concerning this concept are given. Let Δ be a simply connected bounded domain and Φ its boundary; let δ be an open circle and φ its boundary. The notion of a regular homeomorphism between Δ and φ is defined. Every conformal representation is regular and it is shown that the topology of conformal representation can be studied in terms of regular homeomorphisms. A space Y is defined which is made up of the pairs (M, m), where m is the image of M under a regular homeomorphism of Δ into δ. Further results concerning this and related concepts are stated. *D. Montgomery* (Princeton, N. J.).

Choquet, Gustave. Représentation conforme et topologie. C. R. Acad. Sci. Paris **216**, 402–404 (1943). [MF 10021]

With every regular homeomorphism of Δ into δ certain sets A, B and C are defined on φ. The sets A and B are of the third Baire class and can be effectively of this class. On every arc of φ, A contains a perfect set. If the sinuosity of all the accessible points of Φ is uniformly bounded, A is the union of a countable number of perfect sets. The sinuosity of the points of a plane Peanian continuum is uniformly bounded. *D. Montgomery.*

Nöbeling, Georg. Über die topologische Struktur der rektifizierbaren Kontinuen. J. Reine Angew. Math. **184**, 91–115 (1942). [MF 9040]

In this paper topological characterizations are given of the continua of a metric space that are homeomorphic to rectifiable sets (that is, sets of finite Carathéodory linear measure) in a Euclidean space. Relative to sets of a metric space the following definitions are given. (1) A continuum is a compact connected set containing more than one point. (2) By a curve, a rational curve and a regular curve is meant a continuum K each of whose points is contained in a neighborhood of arbitrarily small diameter whose boundary intersects K in a set which contains no continuum, is at most countable or is at most finite, respectively. (3) A continuum K is said to be divided between two of its mutually exclusive closed subsets A and B by a closed set N if $AN=0$, $BN=0$ and $K-N$ is the sum of two sets open in K one containing A and the other B.

The topological characterizations mentioned above are effected by proving the equivalence for a continuum K of the following three propositions. (P_1) K is homeomorphic to a rectifiable set of a Euclidean space. (P_2) K is a regular curve and corresponding to each pair of points p and q of K there exists a natural number n such that p and q are connected by at most n arcs each two having at most a countable set of points in common. (P_3) If A and B are two closed subsets of K such that $AB=0$, then $K-(A+B)$

contains a finite number of noncountable sets $M_1, \cdots, M_m$ with the property that whenever p_k is an arbitrary point of M_k $(k=1, 2, \cdots, m)$ the set $\{p_1\}+\cdots+\{p_m\}$ divides the set K between A and B.

The referee appended some pertinent remarks about the relation of this work and some results of O. G. Harrold and of G. T. Whyburn [see O. G. Harrold, Lectures in Topology, University of Michigan Press, Ann Arbor, Mich., 1941, pp. 237–253; these Rev. **3**, 134]. *J. F. Randolph.*

George, Erich. Wurzeln singulärer Punkte in regulären Kurvenscharen. J. Reine Angew. Math. **184**, 238–252 (1942). [MF 8986]

This paper is a continuation of a former paper by the same author [Deutsche Math. **4**, 462–476 (1939); these Rev. **1**, 29]. The object of the present paper is to investigate singular points of the first type belonging to a regular family of curves. A considerable part of the paper consists of definitions. Regions of a certain kind surrounding a singular point are called "Keile" (wedges). (Distinction is made between "Schleifenkeile," "Strahlenkeile" and "Bogenkeile.") Further sequences of "Keile" called "Keilschachteln" (wedge boxes) are considered. An equivalence notion is defined between "Keilschachteln" of a certain special type. The corresponding equivalence classes are called "Wurzeln" (roots) and their totality receives the name of a "Wurzelkranz" (root wreath). The author defines a linear order in the "Wurzelkranz" and shows that under certain rather complicated conditions the topological structure of the singular point is characterized by the linear order type of the corresponding "Wurzelkranz." *W. Hurewicz.*

Hall, Dick Wick. A note on primitive skew curves. Bull. Amer. Math. Soc. **49**, 935–936 (1943). [MF 9693]

It is shown that a locally connected continuum which is separated by no pair of its points and contains no primitive skew curve of type I (Kuratowski) also contains no primitive skew curve of type II. This result together with a theorem of Claytor is employed to obtain an as yet unpublished characterization of the topological sphere due to F. B. Jones. *G. T. Whyburn* (Charlottesville, Va.).

Tietze, Heinrich. Bemerkungen über verknotete und verkettete Linien. I. Über die speziellen Simony-Knoten und Simony-Ketten. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1942**, 53–62 (1942). [MF 9803]

Reidemeister [Knotentheorie, Ergebnisse der Mathematik, Springer, Berlin, 1932, p. 30] showed that the Minkowski units C_p of the quadratic form f of a knot (or link) are knot (link) invariants for p an odd prime. This paper shows that, if the knot or link is a two-stranded braid of n crossings whose ends are properly connected in pairs, the C_p are easily studied. In fact Reidemeister communicated to the author while the paper was in proof that $f=-nx^2$. The elementary theory of quadratic residues thus makes possible the solution of equivalence problems for many knots and links of this type. The author calls the knots (links) here studied Simony-Knoten (Ketten) in honor of Oskar Simony [Math. Ann. **24**, 253–280 (1884); see also Dehn-Heegaard, Encyclop. Math. Wiss. III A B 3, p. 212].

W. W. Flexner (Ithaca, N. Y.).

Erdös, P. Some set-theoretical properties of graphs. Univ. Nac. Tucumán. Revista A. **3**, 363–367 (1942). [MF 9277]

The author proves the following theorems. (I) Let a and b be infinite cardinals such that $b>a^a$. If the complete graph of power b is split into a sum of a subgraphs, at least one of them contains a complete graph of power greater than a. (II) Assume that the general continuum hypothesis is true, that is, $2^{\aleph_\alpha}=\aleph_{\alpha+1}$. Let $m=\aleph_{\alpha+2}$ and let G be the complete graph of power m. If G is split into subgraphs G_1 and G_2, then either G_1 contains a complete graph of power m or G_2 contains a complete graph of power $\aleph_{\alpha+1}$. *W. Hurewicz.*

Rado, R. Theorems on linear combinatorial topology and general measure. Ann. of Math. (2) **44**, 228–270 (1943). [MF 8284]

The author considers a set A of elements $a_1, a_2, \cdots, a_s$ together with a binary relation $a_i \rightarrow a_j$ subject to no conditions whatsoever. Given a function f on the set A with integer values not less than 0 and given a pair $a_k \rightarrow a_l$ with $f(a_k)>0$, the author defines a new function f_1 by setting $f_1(a_k)=f(a_k)-1$, $f_1(a_l)=f(a_l)+1$, $f_1(a_n)=f(a_n)$ for $n \neq k, l$. The passage from f to f_1 and vice versa is called an elementary transformation. The author states necessary and sufficient conditions under which one function can be obtained from another by a finite succession of elementary transformations. The more general case is considered where the values of the functions f are sets in a space with measure in the sense of the author [Proc. London Math. Soc. (2) **44**, 61–91 (1938)]. *S. Eilenberg* (Ann Arbor, Mich.).

Flexner, William W. Non-commutative chains. II. Ann. of Math. (2) **44**, 628–642 (1943). [MF 9401]

[The first part appeared in Duke Math. J. **8**, 497–505 (1941); these Rev. **3**, 61.] Using noncommutative chains of dimensions 0, 1, 2, 3 in a complex K, a 2-dimensional "homology group" $\mu_2(K)$ is defined. The group of 2-cycles and $\mu_2(K)$ are Abelian. The latter is proved to be subdivision invariant. It is calculated for certain special complexes. No connection is established between $\mu_2(K)$ and the standard groups of K. In every case where $\mu_2(K)$ is calculated, it coincides with the second Hurewicz homotopy group $\pi_2(K)$. It is likewise clear that every 2-cycle is spherical. If, in analogy with paper I, the objective of this paper is to find an algebraic definition of $\pi_2(K)$, it should be remarked that this has been accomplished in another manner. Hurewicz has shown [Nederl. Akad. Wetensch., Proc. **38**, 112–119, 521–528 (1935)] that $\pi_2(K)$ is isomorphic to the second homology group of the universal covering of K; and Reidemeister [Topologie der Polyeder, Akad. Verlagsges., Leipzig, 1938, p. 184] obtained the latter group as the second homology group of K based on "Homotopieketten." *N. E. Steenrod* (Ann Arbor, Mich.).

Montgomery, Deane and Zippin, Leo. A class of transformation groups in E_n. Amer. J. Math. **65**, 601–608 (1943). [MF 9388]

Here G is a compact connected topological group satisfying the second countability axiom which is an effective continuous group of continuous transformations of the Euclidean n-space E and which has at least one $(n-1)$-dimensional orbit. It is known [Montgomery and Zippin, Ann. of Math. (2) **41**, 448–542 (1940); these Rev. **3**, 134; Zippin, Lectures in Topology, University of Michigan Press, Ann Arbor, Mich., 1941, pp. 191–221; these Rev. **3**, 134] that G is a Lie group and that the decomposition space of E whose points are the orbits $G(x)$ of G is homeomorphic to a closed half-line whose end points represent an orbit $G(z)$ of dimension less than $n-1$ and whose other points represent $(n-1)$-dimensional orbits. On this basis it is here

shown: (1) for x and y in $E-G(z)$ the group G_z leaving x invariant is conjugate to G_y; (2) $G(z)$ is a point; (3) there is an arc beginning at z which meets each orbit in exactly one point; (4) if x is not z, the cycles of dimension less than $n-1$ in $E-G(x)$ are bound in $E-G(x)$; (5) for any x the image of a sphere of dimension less than $n-1$ in $G(x)$ can be shrunk to a point in $G(x)$. Number (3) is the principal theorem. It is proved by means of a general lemma on the existence of a cross-sectioning arc in a compact arc-wise connected metric space filled with a continuous collection of compact arc-wise and locally arc-wise connected sets.

W. W. Flexner (Ithaca, N. Y.).

GEOMETRY

Abbott, J. C. The projective theory of non-Euclidean geometry. II. Rep. Math. Colloquium (2) **4**, 22–30 (1943). [MF 8960]

In the most concise foundation of real projective geometry there are two undefined entities, point and line (or point and hyperplane), and two undefined relations, incidence and separation. By increasing the number of assumptions, it is possible to reduce the number of undefined relations to one, that is, to define separation in terms of incidence. (Two pairs of points separate each other if and only if they have no common pair of harmonic conjugates.) This remark emerges from the work of von Staudt, as perfected by Enriques [Lezioni di Geometria Proiettiva, Bologna, 1926].

In the most concise foundation of hyperbolic geometry there is one undefined entity, a point, and one undefined relation, intermediacy (or "betweenness"). The classical procedure of Pasch and Schur enables us to imbed the hyperbolic space in a projective space, and so to use Klein's definitions for congruence and perpendicularity. It was noticed by Menger [Proc. Nat. Acad. Sci. U. S. A. **24**, 486–490 (1938); Bull. Amer. Math. Soc. **44**, 821–824 (1938)] that, by admitting two undefined entities, point and line, it is possible to use incidence instead of intermediacy as the undefined relation. This has the advantage that the two consequent operations of joining and intersecting, interpreted as addition and multiplication, provide an algebraic symbolism which exhibits geometry (either projective or hyperbolic) as a lattice [Garrett Birkhoff, Ann. of Math. (2) **36**, 743–748 (1935).] Menger remarked that it is easy to formulate the necessary assumptions, but admitted that he knew no simple assumptions to that effect. F. P. Jenks [Rep. Math. Colloquium (2) **1**, 45–48 (1939); (2) **2**, 10–14 (1940); (2) **3**, 1–12 (1941); these Rev. **3**, 180] answered that challenge by a set of nine postulates for hyperbolic geometry in terms of incidence. Two of the nine are rather complicated, but the properties of intermediacy and parallelism are deduced without excessive labor. J. C. Abbott [ibid. (2) **3**, 13–27 (1941); these Rev. **3**, 181] showed how pencils of parallels can be interpreted as ends or "points at infinity," but he found it necessary to introduce a tenth postulate for the treatment of congruence. In the sequel now under review, he points out that an eleventh postulate is needed in order to make the locus of ends a conic; it might otherwise be any oval curve. He gives a complicated definition for perpendicularity and deduces its properties very laboriously. [The style is occasionally obscure, as when he writes, on p. 23: "we have to show that Γ' the points P, for any Γ and $Q(\Gamma)$, $Q(\Gamma')$, $R(\Gamma)$, and $R(\Gamma')$ are collinear if they exist."] A simpler procedure, in the reviewer's opinion, would be to define pencils of ultra-parallels as well as pencils of parallels, that is, to define ultra-infinite points as well as ends. We could then imbed the hyperbolic plane in a projective plane before defining congruence and perpendicularity. *H. S. M. Coxeter.*

Davies, E. T. On the isomorphic transformations of a space of K-spreads. J. London Math. Soc. **18**, 100–107 (1943). [MF 9704]

This is a generalization of the theory of affine collineations to K-spreads. Necessary and sufficient conditions for the existence of a continuous group of collineations in a space of K-spreads are given. *M. S. Knebelman.*

Delens, Paul. Sur certaines relations entre tétraèdres et quadriques. J. Math. Pures Appl. (9) **21**, 111–121 (1942). [MF 9299]

Continuing his studies of the geometry of the tetrahedron [J. Math. Pures Appl. (9) **18**, 303–321 (1939); C. R. Acad. Sci. Paris **211**, 220–221, 273–275 (1940); these Rev. **1**, 80; **3**, 86] the author considers systems of desmic tetrahedrons in relation to quadrics and groups of quadrics. He sets up an analytic structure to support and supplement results that have been arrived at before by synthetic considerations, and he hopes that his methods will provide a cohesive principle that will tie together the geometry of the tetrahedron with the geometry of the triangles of its faces. The name of P. de Lépiney [Mathesis **40**, 29 (1926)] should be added to those quoted by the author. *N. A. Court.*

Gambier, Bertrand. Tétraèdres inscrits dans une biquadratique $\mathfrak{B}$ et circonscrits à une quadrique Σ. J. Math. Pures Appl. (9) **21**, 199–265 (1942). [MF 9303]

If the biquadratic $\mathfrak{B}$ is determined by the quadrics Q and Q_1 and if Σ is another quadric, there are six tetrahedra inscribed in $\mathfrak{B}$ and circumscribed on Σ provided the pencil of quadrics $Q+hQ_1$ contains no member cutting Σ in four straight lines. If the pencil contains just one quadric intersecting Σ in four straight lines, there are ∞^1 tetrahedra inscribed in $\mathfrak{B}$ and circumscribed on Σ; their faces envelope one of two developables of class 4 and genus 1 corresponding birationally (plane to point) to $\mathfrak{B}$. If the pencil contains two quadrics each cutting Σ in four straight lines, then each point of $\mathfrak{B}$ is a vertex of ∞^1 tetrahedra inscribed in $\mathfrak{B}$ and circumscribed on Σ; there are, indeed, ∞^2 couples of Möbius; every plane tangent to Σ contains the base of four of the tetrahedra. *J. L. Dorroh.*

Kirsten, Waldemar. Verallgemeinerungen der Geraden-Kugel-Transformation in der Punktreihengeometrie. Deutsche Math. **6**, 409–433 (1942). [MF 8615]

The present paper is based on a series of articles by E. A. Weiss [Deutsche Math. **1**, 23–37, 125–145, 275–290, 447–459, 540–560 (1936); **2**, 171–188 (1937); **3**, 11–35 (1938)]. It begins with a synthetic outline of the line-sphere transformation of ordinary Euclidean [3]. Given two particularized bilinear equations in homogeneous quaternary point coordinates (x), (x'). When the (x') are regarded as fixed, the equations define a line in (x) belonging to a linear line complex. When (x) is fixed, the image line describes a quadratic complex of lines which meet a conic

(the absolute). A special symbolism is now developed and various mappings established, but the essential feature of the proposed generalized line-sphere transformation to [4] is considered in an analytical synopsis as follows: given three particularized bilinear equations, each in five homogeneous point coordinates (x), (x'). When one set remains fixed those of the other define a straight line in the other. It is shown that only four cases need be considered. In the first case the ∞^4 lines in (x), images of the points of (x'), are all secants of a space cubic curve lying in the [3] given by $x_5=0$ (the absolute). If the points (x) of the other space are regarded as fixed, the 10 Grassmann coordinates of the defining line (determinants of order 3 formed from the coefficients of the three linear equations) belong to a pencil of regular tetracomplexes. In the second case, the cubic curve is replaced by a conic, and the Grassmann coordinates in (x) define one regular and one singular tetracomplex. Further specializations exist for the remaining cases.

V. Snyder (Winter Park, Fla.).

Zacharias, Max. Der Caseysche Satz. Jber. Deutsch. Math. Verein. **52**, 79–89 (1942). [MF 9058]

Stated in terms of oriented circles, the theorem of Casey (likewise its converse) holds in complete generality; that is, it is not necessary to exclude those cases in which some of the pairs of circles, whose common tangents are used, have no real common tangent. The orientation determines for each pair whether the common tangent ordinarily called direct or that ordinarily called transverse is to be used. In (1), p. 81, 2_{jk}^2 should be s_{jk}^2. *J. L. Dorroh.*

Tschlek, Jose. Professor Cesáro's method for deriving the formulae of spherical trigonometry. Bol. Mat. **16**, 144–148 (1943). (Spanish) [MF 9919]

Finsterwalder, Sebastian. Die Änderung des Massstabs in einem Dreiecksnetz. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1942**, 21–35 (1942). [MF 9802]

A coordinate net in the neighborhood of a point of a spherical surface is mapped into the plane by each of eight standard map projections. The plane net is then subjected to a uniform stretching and, in each case, mapped back onto the sphere. The resulting unequal expansion of distances and angles on the sphere is examined by the usual method. The author remarks that this process can be extended to a region R of a nonspherical surface by using to approximate R a region of a sphere whose radius is the Gaussian radius of curvature of R at its geocenter. By expanding this sphere a net can be transferred to a region of a different curvature. *W. W. Flexner* (Ithaca, N. Y.).

Hristow, Wl. K. Änderung der geographischen Koordinaten infolge Umorientierung eines geodätischen Netzes und Übergang zum anderen Referenzellipsoid. Z. Vermessungswesen **71**, 132–140 (1942). [MF 9792]

Hristow, Wl. K. Das Wesen der Gauss-Krügerschen Koordinaten. Z. Vermessungswesen **71**, 210–216 (1942). [MF 9796]

Eggert, O. Umformung Soldnerscher Koordinaten in Gauss-Krügersche Koordinaten. Z. Vermessungswesen **71**, 90–98 (1942). [MF 9791]

Werkmeister, P. Zur Ausgleichung des mehrfachen Vorwärtseinschneidens. Z. Angew. Math. Mech. **23**, 118–120 (1943). [MF 9862]

Pinkwart. Zurückführung verschiedener für das Rückwärtseinschneiden angegebener Verfahren auf die Cassinische Lösung. Z. Vermessungswesen **71**, 2–7 (1942). [MF 9788]

Gotthardt, Ernst. Genauigkeitsfragen beim räumlichen Rückwärtseinschnitt und bei der Doppelpunkteinschaltung im Raum. Z. Vermessungswesen **71**, 257–274 (1942). [MF 9797]

Richter, D. H. Bestimmung der ausgleichenden Messungslinie. Z. Vermessungswesen **71**, 7–20, 60–67 (1942). [MF 9789]

Harbert, E. Übersicht der Literatur für Vermessungswesen und Kulturtechnik vom Jahre 1941 mit Nachträgen vom Jahre 1940. Z. Vermessungswesen **71**, 30–56, 78–84 (1942). [MF 9790].

A general bibliography under the following headings: (1) Zeitschriften, Lehr- und Handbücher, sowie grössere Aufsätze, die mehrere Teile des Vermessungswesens behandeln; (2) Mathematik, Tabellenwerke, Rechenhilfsmittel, Physik; (3) Allgemeine Instrumentenkunde, Optik; (4) Höhere Geodäsie und Geophysik; (5) Astronomie und Nautik; (6) Ausgleichungsrechnung und Fehlertheorie; (7) Landesvermessung, Triangulierung und Polygonierung; (8) Stückvermessung, Längenmessung, Flächenmessung, Katasterwesen; (9) Nivellierung, geometrische, trigonometrische, und barometrische; (10) Meteorologie; (11) Tachymetrie, Topographie und Photogrammetrie; (12) Trassieren im Allgemeinen, Abstecken von Geraden und Kurven; (13) Markscheidewesen, magnetische Messungen; (14) Hydrometrie und Hydrographie; (15) Kartographie und Zeichenhilfsmittel; Kolonialvermessungen und flüchtige Aufnahmen; Erdkunde; (16) Geschichte des Vermessungswesens, Landmesservereine und Versammlungen; (17) Organisation des Vermessungswesens, Gesetze und Verordnungen, Unterricht und Prüfungen; (18) Allgemeine Technik, Kulturtechnik, Militärtechnik; (19), Bodenpolitik, Bodenrecht, Rechtskunde; (20) Siedlungstechnik, Umlegung, Volks- und Heimatkunde; (21) Verschiedenes.

Graf, Ulrich. Über das subjektive Erscheinungsbild bei der plastischen Projektion. Z. Instrumentenkunde **63**, 265–275 (1943). [MF 10081]

In a previous paper [Jber. Deutsch. Math. Verein. **52**, 83–95 (1942); these Rev. **5**, 73] the author examined the distortion of a stereoscopic projection due to changes in the position of the observer. In this paper his results are generalized by investigating the relationship between an object and its spatial image, as viewed by an observer in a given position. It is shown that the image may be obtained by subjecting an enlarged (or diminished) model of the object to a perspective collineation. The ratio of enlargement depends on the data of the photograph and the projection only, whereas the collinear transformation depends also on the position of the observer. For one particular enlargement the collineation becomes an affine transformation. *E. Lukacs* (Jacksonville, Ill.).

Convex Domains, Integral Geometry

Dowker, C. H. On minimum circumscribed polygons. Bull. Amer. Math. Soc. **50**, 120–122 (1944). [MF 9896]

Let M_n be a polygon of n sides of least area circumscribed about a convex region R in the plane. The author's first

theorem states that the area of M_n is a convex function of n. The second theorem expresses the corresponding fact for inscribed polygons. The third theorem states that, if R is symmetric about a point O and n is even, then there exists an M_n symmetric about O. The essential tool used in the simple proofs given by the author is formation of new polygons by combining old ones. Thus the proofs of these geometric theorems are largely combinatorial in character.
F. John (Aberdeen, Md.).

de Mira Fernandes, A. Funzioni continue sopra una superficie sferica. Portugaliae Math. **4**, 69–72 (1943). [MF 9374]

Given a positive number λ less than $2\pi/3$ and a bounded closed convex set E in three space, the author finds a rhombus with angles λ at one vertex which circumscribes E. The problem for $\lambda=\pi/2$, due to Rademacher, was solved by Kakutani [Ann. of Math. (2) **43**, 739–741 (1942); these Rev. **4**, 111]. The proof is based on the lemma: for every continuous function on the surface of a sphere in three space and every angle λ less than $2\pi/3$, there are three points whose radii vectors form angles equal to λ and at which the function has the same value. This result for $\lambda=\pi/2$ was given by Kakutani. *L. W. Cohen.*

Santaló, Luis A. A characteristic property of the circle. Math. Notae **3**, 142–147 (1943). (Spanish) [MF 9562]

C. V. Robinson characterized the (full) circle as the only simply connected domain which covers any subset of the plane (by an isometry) provided it can cover each three points of it [Bull. Amer. Math. Soc. **47**, 818–819 (1941); these Rev. **3**, 89]. The author proves the following analogue: if (always by suitable isometries) a circle is contained in each triangle that contains a convex domain, then it will be contained in the domain itself. This property also characterizes the circle. The proof is similar to Robinson's. In the last section related (two-dimensional) problems are discussed. *P. Scherk* (Saskatoon, Sask.).

Santaló, L. A. On the isoperimetric inequality for surfaces of constant negative curvature. Univ. Nac. Tucumán. Revista A. **3**, 243–259 (1942). (Spanish) [MF 9268]

In this paper the author gives a proof, different from those previously published, of the isoperimetric inequality $L^2-F^2-4\pi F\geqq 0$ for surfaces of constant negative curvature, where L is length, F is area, on a surface of curvature -1. The proof stems from the use of a lemma, a generalization of a formula of Poincaré, which states that, for any curve C on the surface, if $n(P)$ is the number of intersections of C with a geodesic circle of radius R and center P, then $\int\int n(P)dP=4L\sinh R$. By considering R as the radius of a geodesic circle which cannot contain C or be contained in C, and in particular the geodesic circle whose area is the same as that of C, the desired inequality is obtained by means of the above integral identity. Uniqueness is also proved by means of a sharpening of the inequality to

$$L^2-F^2-4\pi F\geqq \tfrac{1}{4}F^2(\coth(R_i/2)-\coth(R_e/2)),$$

where R_i is the radius of the smallest circle containing C and R_e is the radius of the largest circle contained in C. In the paper is also to be found a sufficient condition on F and L that a given curve be enclosable in a geodesic circle of radius R. *J. W. Green* (Aberdeen, Md.).

Santaló, L. A. Integral geometry on surfaces of constant negative curvature. Duke Math. J. **10**, 687–709 (1943). [MF 9751]

The author develops the integral geometry in hyperbolic plane in close analogy to the Euclidean case. The following topics are discussed: The measure of straight lines, measure of lines intersecting a curve, Crofton's formula for chords, pairs of straight lines intersecting one another in the interior of an oval, cinematic measure, the isoperimetric inequality and the isoperimetric deficit. Finally, a sufficient condition is given for one oval to contain another (by an isometry).
P. Scherk (Saskatoon, Sask.).

Algebraic Geometry

Segre, B. On ternary non-homogeneous cubic equations with more than one rational solution. J. London Math. Soc. **18**, 88–100 (1943). [MF 9703]

Segre, B. A note on arithmetical properties of cubic surfaces. J. London Math. Soc. **18**, 24–31 (1943). [MF 9207]

The first of these two papers concerns non-singular cubic surfaces F which are rational in the sense that they are definable over the field of rational numbers. The questions treated pertain to the existence of rational points on F and, more generally, of rational lines and of non-trivial rational curves on F. The results are stated without proofs; these will be published elsewhére. Concerning the existence of rational points it is stated that F either carries no rational points or infinitely many. The existence of non-trivial rational curves (that is, curves which are not obtained as the complete intersection of F with another rational surface) is correlated to the existence of a rational line, a rational doublet, triplet or sextuplet of such lines and to the existence of a parametric rational solution of the equation of F. The paper also contains a discussion of the equation $a_1x_1^3+a_2x_2^3+a_3x_3^3+a_4x_4^3=0$ and of special cases of this equation, such as the Reley equation $x_1^3+x_2^3+x_3^3+px_4^3=0$, p not the cube of a rational number. In the second paper the author proves by geometric considerations that a cubic rational surface F with more than one rational point necessarily carries infinitely many rational points. The essential part of the proof is that in which it is shown that, if F carries three collinear rational Eckhardt points, then it carries an infinity of rational points (compare with the next review of the paper by Mordell). *O. Zariski* (Baltimore, Md.).

Mordell, L. J. Segre's indeterminate non-homogeneous cubic equation in three variables. J. London Math. Soc. **18**, 43–46 (1943). [MF 9211]

If a rational cubic surface carries three collinear rational Eckhardt points then its equation can be reduced to the following form: $z(z+2+2px+2qy)=ax^3+bx^2y+xy^2+dy^3$. It has been shown by B. Segre that this equation possesses a parametric solution which involves rational functions of three parameters with rational coefficients [see the preceding review]. In this note this parametric solution is obtained by direct algebraic processes. *O. Zariski* (Baltimore, Md.).

Fischer, I. C. A projective construction for plane nodal cubics. Amer. Math. Monthly **50**, 611–617 (1943). [MF 9852]

The problem considered in this paper is to obtain the elements for the construction of a nodal plane cubic curve

in terms of the ruler alone. It is based essentially on the Grassmann construction and is included as a particular case of a more general one given by W. H. Bunch [Amer. Math. Monthly **42**, 74–80 (1935)]. Consider three lines l_1, l_2, l_3 in a plane forming a triangle and three noncollinear points A, B, C in general position as to the triangle. Let a line through A meet l_1 in x. Connect x with B and C, meeting l_2, l_3 in L', L'', respectively. The line $L'L''$ envelopes a conic, projective with the pencil Ax. The locus of the intersection of corresponding lines of these two pencils, one linear and the other quadratic, generates the desired nodal cubic. A brief synthetic proof is followed by a detailed analytic demonstration, using ABC as reference triangle. Various defining elements are then discussed, distinguishing between the cru-nodal and the ac-nodal cases. *V. Snyder.*

Castrucci, Benedito. On the generation of a plane curve of third order according to the type of Staudt. Anais Acad. Brasil. Ci. **15**, 109–123 (1943). (Portuguese) [MF 9360]

This paper gives a simple characterization for the correspondence between the points of a plane π and their polar conics with respect to a cubic curve lying on π. From this characterization a definition for plane cubics and a way of studying their properties are deduced. Moreover, the author proves Hermite's theorem on nets of conics and adds some trivial remarks about the cubic determinant of order three determined by the coefficients of the general ternary cubic form.

The paper does not achieve the aim declared by the author of obtaining the theory of plane cubics as an extension of Staudt's theory of conics, since the whole treatment is not synthetic but analytic. On the other hand, the same aim was already attained long ago for plane curves of any order by H. Thieme [Z. Math. Phys. **24**, 221, 276 (1878); Math. Ann. **20**, 144 (1882); cf. also J. Rey Pastor, Correspondencia de figuras elementales, Dissertation, Madrid, 1910]. The proof given by the author for Hermite's theorem is incomplete, since he omits to prove that the linear transformation XXVII determined by his equations XXVIII has a nonvanishing determinant. An exactly similar proof was already obtained by H. J. St. Smith [Proc. London Math. Soc. **2**, 85 (1868), §6 or Collected Mathematical Papers, vol. 1, 1894, p. 524]. Hermite's proof is less simple but complete [cf. C. Hermite, J. Reine Angew. Math. **57**, 311 (1860) or Oeuvres, vol. 2, 1908, p. 100]. *B. Segre.*

Derry, Douglas. Affine geometry of convex quartics. Amer. Math. Monthly **51**, 78–83 (1944). [MF 10067]

The author considers a convex curve C given by $f(x, y)=1$, where f is a homogeneous polynomial of degree 4. The problem is to determine among the parallelograms P having one vertex at the origin O and the other three on C the ones with maximum and minimum area. It is shown that either a side or a diagonal of P coincides with one of a pair of "conjugate" diameters of C. If $f(x, y)$ is factored into real quadratic factors $r(x, y)$ and $s(x, y)$ in such a way that the ellipses $r(x, y)=1$ and $s(x, y)=1$ have equal areas, the author defines the intersections of those ellipses as the "critical parallelogram" II. It is shown that all pairs of conjugate diameters are parallel either to the sides or to the diagonals of II. In the case of a maximum P either a side or a diagonal of P coincides with a diagonal of II; in the case of a minimum P either a side or a diagonal of P coincides with a side of II.

[Note of reviewer: "conjugate diameters" are not defined explicitly by the author. For the purposes of the paper they may be taken either as a pair of diameters each of which is parallel to the tangent in the endpoint of the other one, or such that their points at infinity are in involution with the points at infinity of C, paired in any manner. Besides the two real pairs of conjugate diameters used in the paper there should exist an imaginary one.]

F. John (Aberdeen, Md.).

d'Orgeval, Bernard. Remarques sur la détermination des plans multiples représentant une surface algébrique. C. R. Acad. Sci. Paris **215**, 341–342 (1942). [MF 9505]

This note summarizes results obtained by the author in application and extension of the method of Chisini [Accad. Naz. Lincei. Rend. (6) **19**, 688–693, 766–772 (1934); (6) **23**, 22–27 (1936)] for construction of the branch curves (courbes de diramation) of a multiple plane. The method has been used to construct, by projection, the multiple planes corresponding to certain surfaces F, nondevelopable, regular and without singularities except for double curves.

J. L. Dorroh (Baton Rouge, La.).

Gröbner, Wolfgang. Idealtheoretischer Aufbau der algebraischen Geometrie. I. Hamburger Math. Einzelschr. **30**, 56 pp. (1941). [MF 9805]

As the title of this pamphlet indicates, the author sets out to give an exposition of the elementary facts in the theory of algebraic varieties. His method consists in a description of classical geometric results by means of the ideal theory. The projective viewpoint is predominant. Thus, the theory of quasi-divisibility for unmixed $(d-1)$ dimensional ideals in a suitably defined projective integrally closed ring of dimension d is given. The results of these considerations are used to define linear series on a V_d. To obtain numerical invariants of linear series the simple inequalities of the Hilbert function are proved. Since the complete linear series are birationally invariant the author proceeds in the final sections to establish the theorem of Riemann and Roch for curves using the relations on Hilbert functions, though the latter do not give rise to birational invariants. *O. F. G. Schilling* (Chicago, Ill.).

Differential Geometry

Humbert, Pierre. Sur une extension de la notion d'angle: angles d'un faisceau de trois droites. C. R. Acad. Sci. Paris **213**, 970–971 (1941). [MF 9662]

For a concurrent triple of rays in space an angle is defined, which has applications to a geometry in the (x, y) plane with distance from the origin defined by $(x^3+y^3)^{\frac{1}{3}}$. [Cf. same C. R. **211**, 530–531 (1940); these Rev. **3**, 185.]

O. Szász (Cincinnati, Ohio).

Kasner, Edward and DeCicco, John. Union-preserving transformations of space. Bull. Amer. Math. Soc. **50**, 98–107 (1944). [MF 9893]

The authors study transformations in three space of differential elements of order n into lineal elements, the special case when $n=1$. They classify these according to the number of unions converted into unions. Type 3 converts ∞^∞, type 2 converts $\infty^{3(n+1)}$, while type 1 converts all unions. They also investigate union preserving trans-

formations of elements of order n into elements of order m. They show that every such transformation is either an extended point transformation or has $m=1$ and so is the type 1 already discussed. *P. Franklin.*

Segond, Marcel. Aire et congruence de gravité d'une courbe gauche fermée. J. Math. Pures Appl. (9) **21**, 101–109 (1942). [MF 9298]

Let C be a twisted curve. A proof is given by simple vectorial arguments of the fact that there exists a plane curve, the area of whose projection on any plane whatever is equal to that of C. The author admits that this result is probably not new. A study is also made of the "gravitational congruence" of C which is defined as the congruence consisting of all lines forming loci of centers of gravity of plane sections of cylinders passing through C. It is proved that this congruence is algebraic of order not greater than 3 and is linear in the case when C is a skew quadrilateral. Relations are developed between this congruence and the volume obtained by rotating C about an axis.

J. W. Green (Aberdeen, Md.).

Choquet, Gustave. Isométrie des ensembles et cinématique. C. R. Acad. Sci. Paris **214**, 784–786 (1942). [MF 9452]

This paper generalizes the definition of the movement of a plane on a plane. Such a movement is parameterized by a closed set T in a Cartesian space. To every point M of T we make correspond in each of two planes P and Π two half lines d and δ which depend continuously on M. For every point M the mutual positions of P and Π are determined by placing these two half lines in coincidence. Suppose that for each M we make correspond in a continuous fashion points a of P and α of Π. The author defines what is meant by these two image sets being isometric. Under certain conditions he defines a set as being rigid if it is superimposable on every isometric set. He has constructed a Jordan continuum without interior points which is rigid.

D. Montgomery (Princeton, N. J.).

Choquet, Gustave. Isométrie et roulement sans glissement. C. R. Acad. Sci. Paris **214**, 837–839 (1942). [MF 9456]

A movement [see the preceding review] is called a rotation without slipping whenever we may choose the positions of the half lines d and δ in such a way that two conditions are satisfied, one of which is that the positions taken by their extremities a and α are isometric sets. Not every movement is a rotation without slipping. An instantaneous center of rotation and a normal movement are defined and results concerning them are stated. An application is made to geometry. *D. Montgomery* (Princeton, N. J.).

Şemin, Ferruh. Géométrie infinitésimale des systèmes variables à un paramètre. Rev. Fac. Sci. Univ. Istanbul. Ser. A. **6**, 62–82 (1941). (French. Turkish summary) [MF 9741]

Şemin, Ferruh. Géométrie infinitésimale des systèmes variables à un paramètre. Rev. Fac. Sci. Univ. Istanbul. Ser. A. **7**, 20–40 (1942). [MF 9762]

The work as a whole goes back to R. von Mises' "Zur konstruktiven Infinitesimalgeometrie der ebenen Kurven" [Z. Math. Phys. **52**, 44–85 (1905)]. This last was a contribution to kinematic geometry which avoided the usual rigidity assumptions and brought forward the notion of "characteristic" of a variable geometric element (an elaboration of Newton's fluxions). The present paper and its continuation extend these ideas to three dimensions. Noteworthy throughout is the purely geometric treatment with frequent use of detailed diagrams. This first chapter is devoted to characteristics of first order of the fundamental elements (point, line, plane, curve and surface) and to elements defined in terms of these. The characteristic of a point in a one parameter family is a certain point on the tangent line while the characteristic of a plane of a family is a certain plane passing through the characteristic (usual sense) line of the family there. A surface of a family has two types of characteristic, a two parameter family of planes when viewed as a point locus, a surface when viewed as a plane envelope. Similarly a curve of a family can be regarded as a point locus or as the family of its osculating planes; in each case the characteristic is a one parameter family of lines. An exhaustive discussion of the interrelations of these configurations completes the chapter. For example, when a point (of a family) is defined as intersection of curve and surface its characteristic (a point) is intersection of the characteristic of the surface with respect to the point (a plane) and of the characteristic of the curve with respect to the point (a line).

The second chapter is devoted to defining and investigating characteristics of higher order (geometric analogue of higher derivatives) of one parameter geometric systems. For example, the second order characteristic of a point of a curve is a certain point on the principal normal there. Again, the second order characteristic of a curve of a family is a family of hyperbolic paraboloids. Starting with a variable point of a curve the characteristics of first and second order of the many lines and planes associated with the point (normals, rectifying plane, polar axis, etc.) are derived and frequently illustrated with diagrams. Dualizing these configurations leads to a corresponding set of results for a one parameter family of planes. The third chapter applies the previous work to a large variety of geometric problems. For example, by use of characteristics many properties of the circular helix are derived rapidly and synthetically. Here is another typical result: the osculating planes of the trajectories of all points on a tangent sliding on a curve envelope a developable surface of the fourth order and third class. *J. L. Vanderslice.*

Mayer, A. Trajectories on the closed orientable surfaces. Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 71–84 (1943). (Russian. English summary) [MF 8793]

The paper contains the proofs of the nine theorems stated in a previous note [C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 673–675 (1939); these Rev. **2**, 18] concerning the behavior of trajectories determined by a vector field on a closed surface of an arbitrary genus. Of particular interest is the theorem about the decomposition of the surface into regions filled by trajectories of specified types. For the precise formulation of this and other results we refer to the above-mentioned review. *W. Hurewicz.*

Touganoff, N. Sur les surfaces dont la représentation au moyen de normales sur une nappe de leur surface des centres est conforme. C. R. (Doklady) Acad. Sci. URSS (N.S.) **33**, 109–111 (1941). [MF 9624]

By classic methods it is shown that a surface of the type described in the title must necessarily be a surface of revolution. The possible types of surface of revolution are studied and characterizations of them given.

G. A. Hedlund (Charlottesville, Va.).

Liberman, J. Les lignes géodésiques sur une surface convexe de courbure de Gauss positive. C. R. (Doklady) Acad. Sci. URSS (N.S.) **33**, 9–11 (1941). [MF 9617]

In this note the author gives the following result. On a closed surface of everywhere positive curvature, there exists a number N such that every geodesic arc of length at least N must intersect itself at least once. The proof is given by means of a careful study of geodesics and their associated Jacobi differential equations and requires only that the surface be of class $C^{(3)}$. *S. B. Myers.*

Shapiro, J. L. On certain fields of geodesic pencils. C. R. (Doklady) Acad. Sci. URSS (N.S.) **39**, 6–9 (1943). [MF 9575]

A symmetric tensor a_{ij} in a Riemannian space of n dimensions defines a field of pencils of directions, for example, those with components λ^i for which $a_{ij}\lambda^i\lambda^j=0$. If the field has the property that a geodesic tangent to one of the directions remains tangent to the directions of the field, it is called a field of geodesic pencils. The author characterizes stratifiable spaces [Schapiro, same C. R. (N.S.) **32**, 237–239 (1941); these Rev. **3**, 191] by the existence of certain fields of geodesic pencils. For instance, a stratifiable space of order one is characterized by the existence of a field of geodesic pencils defined by $a_{ij}=g_{ij}-\varphi_{,i}\varphi_{,j}$, φ being a scalar function. Various other results are given, among which is one concerning the relation of stratifiable spaces and the space of the gravitational field of a material point.

S. Chern (Princeton, N. J.).

Purcell, Edwin J. Flat space congruences of order one in $[n]$. Trans. Amer. Math. Soc. **54**, 57–69 (1943). [MF 8710]

The author defines an $[n-k]$-congruence of order one in n-dimensional projective space $[n]$ as a linear k-parameter system of $[n-k]$'s in $[n]$ such that one and only one $[n-k]$ of the system passes through an arbitrary point of $[n]$. He derives the properties of these flat space congruences, classifies them and shows that many congruences of order one previously studied are special cases of types in this classification. He notes, however, that certain line congruences treated by Marletta and defined as systems of trisecants or bisecants of given surfaces do not appear in this classification. *T. R. Hollcroft* (Aurora, N. Y.).

Wu, George. Systems of quadrics associated with a point of a surface. I. Duke Math. J. **10**, 499–513 (1943). [MF 8972]

The author defines chord section quadrics $Q^{(1)}$, $Q^{(2)}$ as follows. On an ordinary proper surface S in S_3 choose a curve C and two points x and P on C. Denote the other points of intersection of the two asymptotic curves $u=0$, $v=0$ through each of these points by P_1 and P_2. As P moves along C, the asymptotic chords PP_1 and PP_2 generate, respectively, the ruled surfaces R_1 and R_2. For an arbitrary tangent t_u of S at x, the Moutard quadrics $Q^{(1)}$ and $Q^{(2)}$ of R_1 and R_2, respectively, are the chord section quadrics. The equations of $Q^{(1)}$ and $Q^{(2)}$ are found. Choosing C (1) tangent to $u=0$ at x and (2) tangent to $v=0$ at x, the nonsingular chord section quadrics of C belonging to (1) the u-tangent and (2) the v-tangent, respectively, are in the Darboux pencil. Certain special properties of $Q^{(1)}$ and $Q^{(2)}$ are found when t_u is tangent to C, in particular, a relation between the pencil of chord section quadrics and the Moutard pencil. *T. R. Hollcroft* (Aurora, N. Y.).

Wu, George. Systems of quadrics associated with a point of a surface. II. Duke Math. J. **10**, 515–530 (1943). [MF 8973]

In the paper reviewed above the author defined chord section quadrics and derived some of their properties. He now considers the asymptotic ruled surfaces $R_\lambda^{(u)}$ and $R_\lambda^{(v)}$ generated by the u- and v-tangents along a curve C of a nonruled analytic surface σ and studies projectively covariant figures of σ by means of the Moutard quadrics $Q_u^{(u)}$, $Q_u^{(v)}$ of $R_\lambda^{(u)}$, $R_\lambda^{(v)}$, respectively; $Q_u^{(u)}$ and $Q_u^{(v)}$ are the asymptotic section quadrics belonging to a nonasymptotic tangent t_u of σ. Several theorems concerning the above-defined ruled surfaces and asymptotic section quadrics associated with σ are proved. *T. R. Hollcroft.*

Su, Buchin. The characteristics of asymptotic osculating quadrics of a curve on a surface. Bull. Amer. Math. Soc. **49**, 904–912 (1943). [MF 9688]

The equations and some properties of the asymptotic osculating quadrics of a curve on a surface were derived by E. P. Lane [Bull. Amer. Math. Soc. **33**, 195–200 (1927)]. The present author takes up the investigation using the normal tetrahedron of Cartan and concentrating on the "characteristics" of the two families of quadrics. First the local coordinates of the Cartan reference frames are introduced together with the projective displacement defined by them. In these coordinates the two asymptotic osculating quadrics Q_u, Q_v at any point $P(u, v)$ of the given curve C take relatively simple form and the equations of the characteristics of both families of quadrics are readily derived. At P the characteristic of family Q_u is found to consist of a pair of lines intersecting the asymptotic u-tangent and the latter counted twice; similarly for family Q_v. Three theorems are immediately deduced from these equations. To quote one: if the characteristics of the quadrics Q_u, Q_v along every Darboux curve of a surface S are indeterminate, then S is projectively equivalent to $xyz=1$. Using properties of the characteristics already found, certain covariant configurations at P are derived: a pair of associate directrices, analogues of the directrices of Sullivan, and a pair of associate cones of class three. For example, the u-associate directrix at P is the polar of the tangent plane with respect to a certain covariant trihedron. Each plane of the trihedron is defined as containing the unique curve element of second order in a given Darboux direction whose u-characteristic degenerates into the two asymptotic directions counted twice. The associate cones lead in turn to further configurations: a third cone and a new canonical ray.

J. L. Vanderslice (College Park, Md.).

Wang, Hsien-Chung. On the projective deformation of a family of elements of contact. J. Indian Math. Soc. (N.S.) **7**, 51–57 (1943). [MF 9382]

By an element of contact the author understands a point and a plane element through it. A two parameter family of such elements being given, under what conditions is such a family projectively deformable? If S is the surface of the points, it is obviously necessarily deformable. The author shows that, if S is a nonruled deformable surface, then corresponding to every conjugate net of deformations of S there can be attached to every point of S one and only one element of contact such that the resulting family is deformable. *M. S. Knebelman* (Pullman, Wash.).

Hsiung, Chuan-Chih. An invariant of intersection of two surfaces. Bull. Amer. Math. Soc. **49**, 877–880 (1943). [MF 9682]

If two surfaces S_1, S_2 intersect at an ordinary point $O(0, 0, 0)$ on each, the tangent planes at O being distinct, the equations of the surfaces may be written in the form $y=l_1x^2+m_1z^2+\cdots$, $z=l_2x^2+m_2y^2+\cdots$. The expression $I=l_1^3m_2/(l_2^3m_1)$ is shown to be a projective invariant, and a geometrical interpretation is given of I in terms of a cross ratio. The following expression for I is also found: $I=R_2^4K_2/R_1^4K_1$, wherein K_1, K_2 are the Gaussian curvatures of S_1 and S_2 and R_1, R_2 are the radii of curvature of the curves of section of S_1 and S_2 by the tangent planes to S_2 and S_1 at O. *V. G. Grove* (East Lansing, Mich.).

Hsiung, Chuan-Chih. Projective invariants of a pair of surfaces. Duke Math. J. **10**, 717–720 (1943). [MF 9754]

If two surfaces S_1, S_2 have a common tangent plane at the ordinary points O_1 and O_2 on S_1, S_2, respectively, the coordinate system may be chosen so that the equations of S_1, S_2 may be written in the form

$$x_3/x_4=l_{11}(x_1/x_4)^2+l_{22}(x_2/x_4)^2+\cdots,$$
$$x_3/x_2=l_{33}(x_1/x_2)^2+l_{44}(x_4/x_2)^2+\cdots.$$

The expression $I=l_{11}^3l_{44}/(l_{33}^3l_{22})$ is shown to be a projective invariant, and a geometrical interpretation of I is given in terms of a cross ratio. It is shown that $I=\pm1$ is a necessary and sufficient condition that there exist a projective transformation carrying the elements of the second order of S_1 and S_2 into each other. This transformation depends upon three arbitrary constants. *V. G. Grove.*

Decuyper, Marcel. Sur les couples des surfaces admettant mêmes directrices de Wilczynski. C. R. Acad. Sci. Paris **213**, 428–430 (1941). [MF 9166]

Let (M) be a nonruled surface and $MM_1M_2M_3$ be the normal moving tetrahedron at M, M_3 being its point of intersection with the quadric of Lie, MM_3, M_1M_2 being the directions of Wilczynski. If (P) is another surface having the same Wilczynski directions then the following cases may arise. If $P=M_3$ the two surfaces form a couple of Demoulin-Godeaux. If $P=M+\lambda M_3$ both surfaces have asymptotic lines which are isothermals. Otherwise there are two surfaces one having the Wilczynski directions in the same, the other in the opposite, order and their asymptotic lines belong to linear complexes. *M. S. Knebelman* (Pullman, Wash.).

Wilkins, J. Ernest, Jr. A special class of surfaces in projective differential geometry. Duke Math. J. **10**, 667–675 (1943). [MF 9749]

The paper studies the class of surfaces which satisfy the equation (expressed in the notation of Fubini) (1) $\beta\psi^3-\gamma\phi^3=0$. Many characteristic properties of these surfaces are derived which are expressed in terms of the following projective elements: canonical tangent and line, asymptotic osculating quadric, projective geodesic, pangeodesic, curve of Segre, curve of Darboux, quadric of Darboux. Coincidence surfaces are shown to be a proper subclass of the surfaces satisfying (1) and stronger results are obtained for them. It is also shown that all cubic coincidence surfaces are projectively equivalent. *A. Fialkow.*

Beckenbach, E. F. and Bing, R. H. Conformal minimal varieties. Duke Math. J. **10**, 637–640 (1943). [MF 9744]

The main purpose of this paper is to give a proof promised earlier by the first mentioned author [Duke Math. J. **10**, 335–339 (1943), in particular, p. 336; these Rev. **4**, 278]. This theorem is stated as follows: if the functions $x^i=x^i(u^1, u^2, \cdots, u^n)$, $i=1, 2, \cdots, m$, $m\geqq n\geqq3$, are a set of harmonic functions, that is, $\sum_{j=1}^{n}\partial^2x^i/\partial x^{j2}=0$, $i=1, 2, \cdots, m$, then these functions necessarily are constants or linear functions. Combining this theorem with an extension to varieties of a theorem of Weierstrass on minimal surfaces, the following result is obtained: a necessary and sufficient condition that a minimal variety V_n with $n\geqq3$ in Euclidean m-space E_m can be represented conformally on a domain D in Euclidean n-space is that V_n be contained in a Euclidean subspace E_n of E_m. Conversely, conditions are given that the functions $x^i(u^1, u^2, \cdots, u^n)$ be constant or linear functions. A geometrical characterization of spaces conformally equivalent to Euclidean n-space for $n\geqq4$ has been given by Haantjes [Nederl. Akad. Wetensch., Proc. **43**, 91–94 (1940); these Rev. **1**, 175]. *V. G. Grove.*

Galvani, Octave. Sur la réalisation des connexions euclidiennes ponctuelles à deux dimensions les plus générales. C. R. Acad. Sci. Paris **216**, 23–25 (1943). [MF 9997]

This note is a continuation of previous work of the author [the same C. R. **214**, 337–339, 733–735 (1942); these Rev. **4**, 171, 259]. The main result is based on the observation that, in a Euclidean space of four dimensions, the congruence V of planes parallel to a fixed plane is locally identical to the Euclidean plane E when the planes of V are identified with the points of E. If V is a general congruence of planes, a natural extension of this process will lead to a Euclidean connection. The author proves, using E. Cartan's theory of differential systems in involution, that any given two-dimensional space with a Euclidean connection can be realized in this way. *S. Chern.*

Galvani, Octave. Sur la réalisation des connexions ponctuelles euclidiennes et affines à n dimensions. C. R. Acad. Sci. Paris **216**, 519–521 (1943). [MF 10052]

Continuing the work of previous notes [cf. the preceding review], the author proves that a Euclidean point connection of n dimensions can be realized in a Euclidean space of n^2 dimensions by an n-dimensional variety V_n of plane elements of $n(n-1)$ dimensions (that is, the element consisting of a point and an $n(n-1)$-plane through it). Similarly, an affine point connection of n dimensions can be realized in an affine space of n^2 dimensions by an n-dimensional variety of biplane elements. The latter term means the figure formed by an n-plane and an $n(n-1)$-plane with one point of intersection. *S. Chern* (Princeton, N. J.).

Alardin, Félix. L'autoparallélisme des courbes extrémales dans les espaces métriques fondés sur la notion d'aire. C. R. Acad. Sci. Paris **216**, 277–279 (1943). [MF 10011]

In Cartan's definitive treatment of Finsler space as a space of linear elements endowed with an Euclidean connection the autoparallels and the extremals were identical just as in Riemannian geometry. However, in his corresponding treatment of spaces with an area metric [Espaces métriques fondés sur la notion d'aire, Actual. Sci. Ind., no. 72, Hermann et Cie, Paris, 1933] the two curve families remained distinct. The results stated in the present paper supplement the above monograph. The author succeeds in finding an intrinsic Euclidean connection which retrieves the identity of the two curve families while remaining in the frame of Cartan's work. The condition essential for the complete determination of the desired connection is pre-

sented as a certain restriction on its torsion. The properties of a new second order tensor are especially important for the development. *J. L. Vanderslice.*

Wagner, V. Les espaces de Finsler à deux dimensions à groupes d'holonomie finis et continus. C. R. (Doklady) Acad. Sci. URSS (N.S.) **39**, 210–212 (1943). [MF 10040]

In a Finsler space of two dimensions the group of holonomy is in general an infinite continuous group. The author determined the invariant conditions that this group be a finite continuous group. By the introduction of a direction parameter for the vectors through a point the problem is reduced to that of a finite continuous group of transformations on the parameter. Hence the group of holonomy is of one, two or three parameters. As a consequence of the general results it is seen that the group of holonomy of a Landsberg space is of one parameter. *S. Chern.*

NUMERICAL AND GRAPHICAL METHODS

***Miller, J. C. P. Tables for Converting Rectangular to Polar Co-Ordinates.** Dover Publications, N. Y. (without date). 16 pp. $.75.

These tables facilitate the conversion of rectangular to polar coordinates by means of the equations $r=(x^2+y^2)^{\frac{1}{2}}$, $\theta=\arctan(y/x)$. The argument of the tables is k, where k is the smaller of the two quantities $|x/y|$ and $|y/x|$, so that k lies in the interval 0 to 1. The table then gives the values of $(1+k^2)^{\frac{1}{2}}$ to four decimal places, arctan k and arccot k in degrees to hundredths of a degree, and arctan k in radians to four decimal places, for values of the argument k from 0 to 1 at interval of 0.001. The introduction explains the use of the tables and the determination of the proper quadrant. A supplementary table is provided for reducing to the first quadrant any angle up to thirty-six revolutions. *W. E. Milne* (Corvallis, Ore.).

***Table of Reciprocals of the Integers from 100,000 Through 200,009.** Prepared by the Federal Works Agency, Works Projects Administration; conducted under the sponsorship of the National Bureau of Standards. Technical Director: Arnold N. Lowan. Columbia University Press, New York, 1943. viii+201 pp. $4.00.

This table gives to seven significant figures the reciprocals of the integers from 100,000 through 200,009. It is intended to supplement available tables for 7-figures reciprocals (for example, those of Oakes or of Cotsworth) in the region where interpolation in the latter is difficult. To facilitate interpolation in the present tables, each page has a small table of proportional parts applicable to the differences on that page. *W. E. Milne* (Corvallis, Ore.).

***Table of Circular and Hyperbolic Tangents and Cotangents for Radian Arguments.** Prepared by the Federal Works Agency, Work Projects Administration; conducted under the sponsorship of the National Bureau of Standards. Technical Director: Arnold N. Lowan. Columbia University Press, New York, 1943. xxxviii+410 pp. $5.00.

The four functions are tabulated in parallel columns. The tabular interval in the main table is .0001. The range and the number of significant (S) or decimal (D) figures are as follows: $\tan x$ and $\tanh x$ [0, .01; 8S], [.01, .1; 9D], [.1, 2; 8S]; $\cot x$ [0, .1; 8D], [.1, 1.5700; 8S], [1.5700, 1.5750; 13D], [1.5750, 2; 8S]; finally $\coth x$[0, .1; 8D], [.1, 2; 8S]. A second table gives ten decimal places of all functions for the range [0, 10] in steps of .1. Second central differences are given where they are of practical use. Valuable are auxiliary tables for interpolation giving, in steps of .001, values of $p(1-p)/2$ (exact) and of $p(1-p^2)/6$ (to six decimal places). The book includes furthermore conversion tables for radians and degrees, and multiples of $\pi/2$. *W. Feller* (Providence, R. I.).

***Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments.** Prepared by the Federal Works Agency, Work Projects Administration; conducted under the sponsorship of the National Bureau of Standards. Technical Director: Arnold N. Lowan. Columbia University Press, New York, 1943. xliv+403 pp. $5.00.

The high standard of the earlier W.P.A. tables is fully maintained in this new volume, in which the real and imaginary parts of $J_0(\rho e^{i\varphi})$ and $J_1(\rho e^{i\varphi})$ are tabulated to ten decimal places for the ranges: ρ, 0(.01)10; φ, 0°(5°)90°. Tables of five point Lagrangian interpolation coefficients are also given to ten decimal places for the ordinates 0(.001)1. These may be used satisfactorily to interpolate for values of ρ along any tabulated φ-ray, but less accuracy is obtainable in interpolating for nontabulated values of φ. In the introduction there is a brief discussion of the use of Bessel functions of complex argument in physics and engineering. Then the formulas used in the computations and the various checks employed at every stage are fully described. *M. C. Gray* (New York, N. Y.).

Blanch, G. and Rhodes, I. Seven-point Lagrangian integration formulas. J. Math. Phys. Mass. Inst. Tech. **22**, 204–207 (1943). [MF 9778]

The authors present a number of formulas for numerical integration of $f(x)$ over an extended region when the values of $f(x)$ are known at equally spaced points with interval h throughout the region. Typical of these (and the most useful) is the following, valid for $s\geqq 14$:

$$\int_{x_1}^{x_1+(s-1)h} f(x)dx=\frac{h}{120960}[36799(f_1+f_s)+176648(f_2+f_{s-1})+54851(f_3+f_{s-2})+177984(f_4+f_{s-3})+89437(f_5+f_{s-4})+130936(f_6+f_{s-5})+119585(f_7+f_{s-6})]+h(f_8+f_9+\cdots+f_{s-7})+R,$$

where $|R|<0.02(s-1)\cdot|h^9f^8(\xi)|$. (If $s=14$, the set of terms $h(f_8+f_9+\cdots+f_{s-7})$ should be dropped.) Other formulas similar in spirit and accuracy are given for the cases $s\geqq 19$ odd, $s\geqq 22$ even, $s=8$, $s=10$, $s=12$, $s=13$. *W. E. Milne* (Corvallis, Ore.).

Samuelson, Paul A. A simple method of interpolation. Proc. Nat. Acad. Sci. U. S. A. **29**, 397–401 (1943). [MF 9771]

To obtain the polynomial $y=a_0+a_1x+\cdots+a_nx^n$ satisfied by the values (x_0, y_0), $\cdots$, (x_n, y_n) the author constructs the set of $n+2$ polynomials

$$P_{n-k}(x)=\sum_{i=0}^{n-k} b_ix^{n-k-i},\quad k=-1, 0, 1, \cdots, n+1,$$

where $b_0=1$ and $P_{n+1}(x)$ vanishes for $x_0, x_1, \cdots, x_n$. Then the coefficient a_j in the desired polynomial is given by

$$a_j=\sum_{i=0}^{n} y_i \frac{P_{n-j}(x_i)}{P'_{n+1}(x_i)}.$$

Convenient methods of performing the calculations are indicated. [Reviewer's note: The paper contains troublesome errors. As it stands, the fundamental equation (1) is entirely false and the matrix multiplication in the illustrative example is wrong.] *W. E. Milne.*

Heinhold, J. Zur Interpolation bei ungleichen Tafelabständen. Z. Angew. Math. Mech. **22**, 235–238 (1942). [MF 8921]

The author writes the general Newton interpolation formula for unequal intervals of the argument in a form strictly analogous to the usual form for equal intervals. This is accomplished by merely multiplying and dividing by appropriate factors. The resulting expression has the advantage that the absolute value of any term is less than the absolute value of the coefficient, a generalized difference, provided the position of the interpolated value is appropriately restricted. Formulas are given for the calculation of the coefficients and the other parameters involved.

P. W. Ketchum (Urbana, Ill.).

Spoerl, Charles A. Difference-equation interpolation. Trans. Actuar. Soc. Amer. **44**, 289–325 (1943). [MF 9739]

This paper is devoted primarily to a discussion of osculatory interpolation formulas suitable for extended subtabulation in a table or series of values with equally spaced entries. Particular attention is given to Henderson's four-point formula, in which consecutive arcs have the same tangents and the same curvatures at the junction points. In the sequence of formulas for consecutive arcs the set of coefficients of the higher degree terms satisfies a second order linear difference equation whose solution involves two undetermined constants. The author investigates in detail the effect on the interpolated values of various proposed ways of choosing these two constants and concludes that an excellent way is to minimize the sum of the squares of the second differences of the interpolated series.

The paper goes on to adapt the Whittaker-Henderson graduation formula B to the problem of interpolation, with or without graduation of the original values. If there is no graduation, this gives a result very similar to the Henderson formula. For the case of both graduation and interpolation, there arises the problem of how much weight to give to the graduation.

A long example is worked by five different methods, with the true values for comparison. An appendix supplies auxiliary tables, goes into the details of the solution of the difference equation and indicates suitable computational procedure. *W. E. Milne* (Corvallis, Ore.).

Popov, A. A. A new method for graphoanalytical integration. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 286–288 (1943). [MF 9577]

A method is given for finding the moment $I=\int \phi(x)f(x)dx$ of a graphically given function $f(x)$ with respect to the analytical weighting factor $\phi(x)$. The range of x is first subdivided and $f(x)$ approximated in each subdivision by a straight line segment. Given a $\phi(x)$, the moment I and center of gravity of the resulting trapezoidal pieces is calculated analytically, once for all. A simple graphical construction, based on what the author calls the "singular focal point," is used to combine the moments of these pieces. The singular focal point is the point with the same abscissae as the center of gravity and ordinate equal to I/A, where A is the area under the curve $\phi(x)$. The singular focal point of the sum of two of the trapezoidal pieces is collinear with the singular focal points of the separate trapezoids.

P. W. Ketchum (Urbana, Ill.).

Marchant Methods. Milne method of step-by-step integration of ordinary differential equations when starting values are known. MM—216. 10 pp. [MF 10193]

Marchant Methods. Milne method of step-by-step double integration of second order differential equations in which first derivatives are absent. MM—216A. 6 pp. [MF 10194]

Marchant Methods. Starting values for Milne-method integration of ordinary differential equations of the first order. The method of Milne. MM—260. 11 pp. [MF 10195]

Marchant Methods. Starting values for Milne-method integration of ordinary differential equations of first order, or of second order when first derivatives are absent. The method of Taylor's series. MM—261. 4 pp. [MF 10196]

Detailed description of the most practical computational arrangement for the well-known Milne method. The theoretical foundations of the latter are also described, and appraisals of the remainder terms are given. The pamphlets are published and distributed by the Marchant Calculating Machine Co., Oakland, Calif.

Stohler, Karl. Eine Vereinfachung bei der numerischen Integration gewöhnlicher Differentialgleichungen. Z. Angew. Math. Mech. **23**, 120–122 (1943). [MF 9861]

Condensed forms are derived for standard successive approximation numerical integration formulas for first and second order ordinary differential equations as presented by H. v. Sanden in "Praxis der Differentialgleichungen" [Berlin, 1943]. *N. A. Hall* (New Haven, Conn.).

Bückner, Hans. Eine vom Vorhaltschiessen abgeleitete Interpretation einer Näherungslösung der gewöhnlichen linearen Differentialgleichung 1. Ordnung. Z. Angew. Math. Mech. **23**, 122–123 (1943). [MF 9860]

A gun placed at O, the origin of coordinates $(0, 0)$, is firing on a target moving along the line $y=k$ $(k>0)$. When the moving target is at point A on line $y=k$, the gun is fired. The clock time of this firing is denoted by t_A. The bullet is assumed to hit the target at point T on $y=k$. Let $\lambda(t_A)$ and $-\mu(t_A)$ represent the time of flight of a bullet from O to T and from O to A, respectively. Since $t_A+\mu(t_A)$ represents the clock time of firing the bullet traveling from O to A, we have

$$\lambda(t_A+\mu(t_A))=-\mu(t_A).$$

Replacing t_A by x in the above equation, we obtain

$$\lambda(x+\mu)+\mu=0. \tag{1}$$

Equation (1) is the same functional equation used by the author [Z. Angew. Math. Mech. **22**, 143–152 (1942); these Rev. **5**, 6; equation (1) in this review should read $y+\lambda y'=f(x)$] to obtain an approximate solution

$$u=f(x+\mu(x)) \tag{2}$$

to the differential equation

$$u+\lambda(du/dx)=f(x).$$

Based on the correspondence between (1) and (2), the author states that, if $f(t_A)$ gives the position of the target when struck by the bullet, the approximating function $f(t_A+\mu(t_A))$ gives the position of the target at the time (t_A) the bullet was fired. *F. G. Dressel.*

Babakov, I. M. Zur Berechnung der höheren Eigenfrequenzen der Drehschwingungen von reduzierter Welle. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 109–124 (1941). (Russian. German summary). [MF 7702]

A shaft, subdivided in n sections of torsional stiffness c_i, carries $n+1$ discs. The vibrating system under consideration has obviously n essential degrees of freedom. The author denotes by $l_1, \cdots, l_n$ the amplitudes of the periodic torques acting on the n sections of the shaft, the frequency of the forced oscillations being a. Let $\lambda_{s1}, \cdots, \lambda_{sn}$ $(s=1, \cdots, n)$ be the normal modes of the free oscillations corresponding to the eigenfrequency p_s. The amplitude λ_i of the forced oscillation is determined by the well-known formula

$$\lambda_i=\sum A_{ij}l_j, \quad A_{ij}=\sum_{k=1}^{n}\lambda_{ki}\lambda_{kj}/(p_k^2-a^2).$$

It is known that the normal modes are the solutions of the homogeneous equations

$$(*) \qquad \lambda_i=(p^2-a^2)\sum A_{ij}\lambda_i$$

corresponding to the eigenvalues $p_s^2-a^2$. By a suitable choice of the parameter a, $p_s^2-a^2$ becomes, for any value of s, the smallest eigenvalue (in absolute value) of (*) and can be approximately computed by any method for the determination of the absolute smallest eigenvalue. In this way every frequency $p_1, \cdots, p_n$ can be computed separately and independently. The author gives several numerical applications. *A. Weinstein* (Toronto, Ont.).

Southwell, R. V. and Vaisey, Gillian. Relaxation methods applied to engineering problems. VIII. Plane-potential problems involving specified normal gradients. Proc. Roy. Soc. London. Ser. A. **182**, 129–151 (1943). [MF 9775]

The relaxation methods, which have been applied by Southwell and his pupils to the solution of two-dimensional potential problems, are applied in this paper toward the determination of harmonic functions which satisfy mixed boundary conditions involving values of the function as well as of its gradient at the boundary. *H. Poritsky.*

van Reijen, L. L. Diffraction effects in Fourier syntheses and their elimination in X-ray structure investigations. Physica **9**, 461–480 (1942). [MF 9818]

The distribution of electron density over a crystal face has a formulation as a double Fourier series of which the leading coefficients may be determined by intensity measurements of X-ray reflections. The Gibbs phenomenon is evident if the distribution is synthesized with the unknown coefficients taken zero. The author proposes a simple way of extrapolating values for the missing coefficients so as to eliminate the meaningless diffraction ripples.

R. L. Dietzold (New York, N. Y.).

Hansel, C. W. Standard curves. Philos. Mag. (7) **34**, 361–376 (1943). [MF 8733]

The quadratic function $y=x^2-2px$ is calculated from the Cartesian graph of the "standard curve" with $p=1$ and a change of scale factor on the x and y axes. Numerical examples are given of this and similar procedures on nomograms for x^3-3px, $\sin px$ and $\exp(-p^2x^2)$.

P. W. Ketchum (Urbana, Ill.).

Samuelson, Paul A. Efficient computation of the latent vectors of a matrix. Proc. Nat. Acad. Sci. U. S. A. **29**, 393–397 (1943). [MF 9770]

The author states that previously known algebraic methods for computing the latent vectors of an n-rowed square matrix a involve multiplications which increase with the fourth power of n. He then presents a procedure involving approximately $8n^3/3$ multiplications. Let h be an arbitrary column vector. Then the coefficients in the characteristic equation $\sum_{i=0}^{n}p_ix^{n-i}=0$ are found by solving the n linear equations $\sum_{i=0}^{n}(a^{n-i}h)p_i=0$. Let $u_{kj}=\sum_{i=0}^{n-k}p_ix_j^{n-k-i}$, $k=0, 1, \cdots, n-1$, where x_j are the n roots (assumed distinct) of the characteristic equation. Then the required latent vectors are the columns of the matrix product of the n-rowed square matrix a^ih $(i=0, 1, \cdots, n-1)$ and the n-rowed square matrix u_{kj}. The proof is conducted by means of the known solution of the nth order differential system $Dy(t)=ay(t)$, $y(0)=h$. [Misprint in first equation of paper: instead of p_{n-1} read p_n, etc.] *W. E. Milne.*

Spoerl, Charles A. A fundamental proposition in the solution of simultaneous linear equations. Trans. Actuar. Soc. Amer. **44**, 276–288 (1943). [MF 9738]

The solution of the n linear equations $Mu=w$, where M is a symmetrical square matrix, u is an unknown vector and w a given vector, can be carried out through the step-by-step calculation of an intermediate vector v, a set of n quantities σ_i and a triangular n-rowed matrix A with unit elements in the principal diagonal and zeros above that diagonal. Let B denote the triangular matrix obtained by dividing the ith row of the transpose of A by σ_i. Then A and the σ_i are found from $AB=M$, the vector v is obtained from $w=Av$ and finally u is found from $v=Bu$.

The author points out that for the general case the above procedure offers nothing new and that other methods are more direct. However, in the case where n is large and all elements of M beyond a certain distance from the principal diagonal vanish, this method is very effective. Such cases arise, for example, in the solution of linear difference equations. The author works out in detail a number of examples, giving practical suggestions for the best arrangement of the work. The general solution for the case of arbitrary w is explained, as well as cases where the diagonal of M is infinite in one direction or in both directions.

W. E. Milne (Corvallis, Ore.).

Müller, Edgar. Genauigkeit der bei Reduktion von Fehlergleichungen eliminierten Unbekannten. Z. Vermessungswesen **71**, 186–190 (1942). [MF 9794]

In this paper observational equations are considered of the type $a_ix+b_iy+c_iz=l_i$, where the a_i's are unity and the b_i's, c_i's and l_i's are known. The normal equation for x allows x to be eliminated from the observational equations; then the normal equations in y and z in the resulting two-unknown problem are identical with the "first reduced set" of normal equations in the ordinary Gaussian solution of the three-unknown problem. The paper is concerned with the estimation of the standard deviation of the least squares value of the eliminated x, and special formulae are derived

from the conventional solution of the three-unknown problem.

The reviewer points out that the reduced observational equations are the conventional equations

$$(b-\bar{b})y+(c-\bar{c})z=l-\bar{l}$$

(bars denoting mean values) of multiple regression, in which the variates b, c and l are measured from their means.

T. E. Sterne (Aberdeen, Md.).

Wittke, H. Doppelrechenmaschinen. Z. Instrumentenkunde **63**, 396–398 (1943). [MF 10084]

Werkmeister, P. Eine neue Bauart der Rechenmaschine Mercedes-Euklid. Z. Instrumentenkunde **63**, 359–361 (1943). [MF 10085]

Fehrentheil und Gruppenberg, Ladislaus Ritter von. Vereinfachte Quadratwurzelziehung mit der Rechenmaschine. Z. Instrumentenkunde **62**, 227–230 (1942). [MF 9883]

Wiseman, Robert T. The use of punched card equipment for the calculation of policy values and guarantees. Trans. Actuar. Soc. Amer. **44**, 326–342 (1943). [MF 9740]

MATHEMATICAL PHYSICS

Seemann, H. Die Regelflächen der Raumgitterinterferenzen und ihre wichtigsten ebenen Schnitte als Einkristalldiagrammkurven. Z. Phys. **119**, 374–396 (1942). [MF 9413]

The paper deals with the reflection of monochromatic X-rays on the surface of a crystal. A small diaphragm is placed at a certain distance above the crystal and selects a solid cone of X-rays diverging from an extended target. Of these rays only a one-dimensional manifold is reflected and forms, after reflection, a ruled surface of special type. In a previous paper [Ann. Physik (5) **7**, 633–648 (1930)] the author has discussed this type of surface and its cross sections with certain planes (photographic films). In the present paper the same results are derived by elementary geometric methods. *R. Luneberg* (Buffalo, N. Y.).

Schulz, H. Beiträge zur geometrischen Optik. Z. Instrumentenkunde **62**, 119–122 (1942). [MF 9969]

A method in the design of simple achromates is developed which permits advance determination of the glass types in such a way that the Seidel conditions $S(1)$ and $S(2)$ and the condition for cementing are satisfied. *P. Boeder.*

Novobatzky, K. F. Lichtbeugung an schwarzen Schirmen. Z. Phys. **119**, 102–113 (1942). [MF 9415]

The approximate treatment of diffraction by a screen, based on the Kirchhoff integral, is improved upon and given a form from which the state of polarization of the diffracted light can be determined. *H. Poritsky.*

Ramachandran, G. N. The theory of coronae and of iridescent clouds. Proc. Indian Acad. Sci., Sect. A. **17**, 202–218 (1943). [MF 9266]

Ramachandran, G. N. On the transmission of light through a cloud of randomly distributed particles. Proc. Indian Acad. Sci., Sect. A. **17**, 171–186 (1943). [MF 10108]

In these papers the author develops a theory of coronae based on the transmission of light through a cloud of transparent droplets. The wave amplitude in the direction of the incident ray is first evaluated, and then approximate values are found for the wave amplitude in other directions. The directions for maxima and minima of the resulting intensities are discussed, and it is shown that a small variation in the size of the droplets produces a large change in the distribution of the light and dark rings in the corona. This is true until the radius of the drops becomes large compared with the wavelength when the cloud behaves substantially like an opaque disc. *M. C. Gray* (New York, N. Y.).

Bouman, J. A general theory of lattice-distortions. Physica **9**, 29–49 (1942). [MF 9808]

An expression for the intensity of a diffracted beam of X-rays is derived for a distorted crystal. This formula includes the results of Dehlinger and Boas and the intensity caused by the distortion, which is known as the frozen heat motion. A theorem, valid for all inhomogeneous distortions, is proved with the help of this expression. This theorem can be used for the discussion of the broadening or decrease of intensity of the reflection for distorted crystals.

Author's summary.

Wang, J. S. Note on Kirkwood's theory of superlattices. Acad. Sinica Science Record **1**, 116–120 (1942). [MF 8846]

Let the sites which can be occupied by atoms be divided into two classes C and D each containing N members. Let ξ_a denote the state of occupation of a site a of C by an atom, being 1 for an A atom and 0 for a B atom. Let μ_{ab} be 1 when a and b are neighboring sites but otherwise zero. Let ζ_a denote the number of A atoms in sites next to a, so that $\zeta_a=\sum_{b=1}^{N}\mu_{ab}\eta_b$, where η_b is the state of occupation of a site b of D by an A atom. The aim is to calculate the moments of the quantity

$$X=\sum_{a=1}^{N}\xi_a\zeta_a=\sum_{a,b=1}^{N}\mu_{ab}\xi_a\eta_b.$$

Kirkwood used the first form to calculate moments up to the third order. Later he and Bethe carried the calculations up to the fourth order with the aid of the second form for X. Wang now shows that the first form can be used for the exact evaluation of moments up to any order and that, if ξ, ξ', $\cdots$, $\xi^{(m-1)}$ are m various values of ξ_a, n the number of A atoms in C, then

$$\overline{\xi\xi'\cdots\xi^{(m-1)}}=(-n,m)/(-N,m),$$

where $(p,m)=p(p+1)\cdots(p+m-1)$. It is found that, when surface effects are neglected, careful consideration should be given to the order of terms in the expression

$$\log\left(\sum_{r=0}^{\infty}M_r x^r/r!\right)=\sum_{r=0}^{\infty}\lambda_r x^r/r!,\qquad M_r=\overline{X^r},$$

which is usually used for the calculation of the partition function in the statistical theory of superlattices.

H. Bateman (Pasadena, Calif.).

v. Laue, M. Eine Ausgestaltung der Londonschen Theorie der Supraleitung. Ann. Physik (5) **42**, 65–83 (1942). [MF 9237]

The author shows that the system of equations for the electrodynamics of the superconducting state:

$$c\,\mathrm{curl}\,\lambda J=-H,\quad(\partial/\partial t)(\lambda J)=E,$$

where E is the electric and H the magnetic field strength, J the current density, λ the characteristic constant of the superconductor in question, can be written in a Lorentz-invariant form by introducing the four vector "Viererleitung" which is defined, in the system in which the superconductor is at rest, by the components $J_4=0$; $J_1, J_2, J_3=J$. [This property does not characterize the above equations among other formulations of the theory which have been discussed in the literature.] For the case that normal conducting electrons are present in the superconductor, a slightly modified form of the original theory is proposed in which the electric charge density ρ satisfies the equation

$$\ddot{\rho}+\sigma\dot{\rho}+(1/\lambda)\rho=0,$$

σ being the electric conductivity. The theory is extended to the case in which λ depends on temperature, time- or -space coordinates. *F. W. London* (Durham,N. C.).

Potier, Robert. Sur le carré moyen de la distance parcourue par des neutrons lents dans un milieu ralentisseur. C. R. Acad. Sci. Paris **215**, 270–271 (1942). [MF 9497]

Sauter, Fritz. Zur Theorie der metallischen Elektrizitätsleitung. Ann. Physik (5) **42**, 110–141 (1942). [MF 9238]

The author objects to the Bloch-Peierls electronic theory of metals in that it uses characteristic functions which are valid only for the ideal crystal but not for the real crystal nor for liquid metals. He therefore prefers to develop instead a theory on the basis of a model which uses as first approximation simple plane waves and considers all interactions between electrons and lattice simultaneously as scattering processes of the plane waves on the vibrating lattice. The method proposed may be described as a kind of generalization of the theory of Houston [Z. Phys. **48**, 444 (1928)]. The condition of stationary equilibrium is essentially identical with the integral equation of Bloch. Accordingly the results are so far essentially the same as in Bloch's theory. *F. W. London* (Durham, N. C.).

Picht, Johannes. Beiträge zur Theorie der elektrischen Ablenkung von Elektronenstrahlenbündeln. IV. Ann. Physik (5) **43**, 53–72 (1943). [MF 9929]

Brillouin, Leon. The antenna problem. Quart. Appl. Math. **1**, 201–214 (1943). [MF 9362]

The antenna problem is defined as the determination of the current distribution on a wire of finite radius subject to free or forced vibrations. The author formulates an integro-differential equation applicable to a perfectly conducting cylindrical wire on the assumptions that the current is equally distributed around the cylinder and that it flows along the surface. This equation can be separated into an integral equation and a differential equation, and, if the solution of the integral equation can first be found, then the differential equation can be solved by successive approximations to give the actual current distribution. Unfortunately the author does not give an actual solution of the integral equation so that from the practical point of view he leaves the problem unsolved. *M. C. Gray.*

Bouwkamp, C. J. Hallén's theory for a straight, perfectly conducting wire, used as a transmitting or receiving aerial. Physica **9**, 609–631 (1942). [MF 9822]

The theory of Hallén [Uppsala Universitets Årsskrift 1930, no. 1; Nova Acta Uppsala (4) **11**, no. 4 (1933)] of general "one-dimensional" transmitting and receiving aerials differs from elementary theories in that it does not start from a given current distribution but determines it and the antenna field from Maxwell's equations. All quantities of the order a/l or higher ($2l$=length, $2a$=diameter of antenna wire) are neglected, and the results appear as expansions in terms of Ω^{-1}, where $\Omega=2\cdot\ln(2l/a)$. This is a large quantity and only the term with Ω^{-1} is explicitly obtained. No formal proof of convergence is given. This theory is here simplified and carried further by restricting it to the case of a straight wire of infinite conductivity and uniform cross-section; furthermore, the approximation for the middle (feeder connection point) of the aerial, that is, in the expressions for the radiation resistance and reactance, is extended to terms of order Ω^{-2}. The results are presented in form of tables and curves and are illustrated by numerical examples. The numerical results of the second-order approximation seem to indicate that convergence (or the physical significance of higher terms in Ω^{-1} in case of semi-convergence [which seems more probable to the reviewer]) is doubtful for $\Omega\lesssim 10$ and $(2\pi l/\lambda)\gtrsim 5$. *H. G. Baerwald.*

Sommerfeld, A. und Renner, F. Strahlungsenergie und Erdabsorption bei Dipolantennen. Ann. Physik (5) **41**, 1–36 (1942). [MF 9229]

This is a more detailed version of a paper with the same title published by the authors in Hochfrequenztech. Elektroak. **59**, 168–173 (1942) [cf. these Rev. **4**, 287]. Complete derivations of the formulas for the radiated power are included. *M. C. Gray* (New York, N. Y.).

Carter, P. S. Antenna arrays around cylinders. Proc. I. R. E. **31**, 671–693 (1943). [MF 9671]

Niessen, K. F. Mittlere Frequenzstabilität von Hohlräumen. Physica **9**, 145–157 (1942). [MF 9812]

The stability of the gravest natural frequencies of electromagnetic oscillations in metal enclosed hollows of the shape of a cube or sphere is computed for small dilatations. The author takes into account the change of triple into double degeneracy by considering the weighted mean values of the frequency shifts of the gravest oscillation of each mode (only one of which remains the absolutely gravest eigenfrequency). It is then found that, for deformations leaving the surface area invariant, cube and sphere have, in first approximation, nearly equal frequency stability. *H. G. Baerwald* (Cleveland, Ohio).

Niessen, K. F. Über die Frequenzbeständigkeit einiger Hohlräume in einem elektrischen Kreis. Physica **9**, 539–546 (1942). [MF 9821]

It is argued that for applications to frequency control at ultra high frequencies the frequency stability of metal enclosed hollows should be measured by mean values of the absolute frequency shifts (weighting according to the degeneracy of the modes) rather than by means of their algebraic values as proposed in the paper reviewed above. *H. G. Baerwald* (Cleveland, Ohio).

LePage, Wilbur Reed. A method for determining the normal modes of Foster's reactance networks. Elec. Engrg. **62**, 674–678 (1943).

Based upon the fact that the network function of a Foster reactance network is a rational function with alternate zeros and poles, all of which are simple, the author deduces approximation methods to find the zeros between successive

poles and demonstrates their rapid convergence in special cases. *E. Weber* (Brooklyn, N. Y.).

Nijenhuis, W. and Stumpers, F. L. On some properties of electrical networks. Physica **8**, 289–307 (1941). [MF 9544]

This paper contains a development of the familiar relations giving an admittance or impedance function of a realizable network in terms of its real or imaginary part or its attenuation or phase characteristic. A study is also made of possible transfer impedances with constant phase or constant attenuation. *C. E. Shannon.*

Nijenhuis, W. On transients in homogeneous ladder networks of finite length. Physica **9**, 817–831 (1942). [MF 9826]

Solutions are obtained for transients in a finite ladder electrical network analogous to the d'Alembert-Euler and the Bernoulli solutions of a stretched string. The low pass and diffusion cases are treated in detail and from the former an approximation to Bessel functions is obtained for the range where the argument is near the order.

C. E. Shannon (New York, N. Y.).

Parodi, Maurice et Raymond, François. Application de la fonction U_n de Lucas au calcul des fréquences propres d'ensembles de circuits couplés comportant des relais électroniques. C. R. Acad. Sci. Paris **215**, 459–460 (1942). [MF 9512]

It is shown that the solution of filter circuit equations for the currents in a multiple electronic relay circuit can be expressed in terms of the functions of Lucas. [Note by reviewer. This result is apparently due to Lagrange; see, for instance, Rayleigh, "Theory of Sound," vol. 1, Macmillan, London, 1927, page 172.] *H. Poritsky.*

Stockman, Harry. A treatment of non-linear devices based upon the theory of related linear functions. J. Appl. Phys. **14**, 645–658 (1943).

Very often electrical devices containing decidedly non-linear elements are so designed that their overall operation is linear. When this is the case, overall relationships are given by linear algebraic equations. *N. Levinson.*

Bennett, W. R. Response of a linear rectifier to signal and noise. J. Acoust. Soc. Amer. **15**, 164–172 (1944). = Bell System Tech. J. **23**, 97–113 (1944). [MF 9855]

Approximate solutions are found for the low frequency spectrum of the output of a linear rectifier whose input is a sinusoidal signal plus a narrow band noise of high frequency, as in ordinary envelope detection. A simpler derivation of the d.c. component is also given. *C. E. Shannon.*

Surdin, M. Distribution in time of spontaneous fluctuation voltage. Philos. Mag. (7) **34**, 716–722 (1943). [MF 9560]

The known probability distribution functions for amplitude of fluctuation voltages and for amplitudes after linear and square law rectification are derived by loose thermodynamic arguments. It is stated that linear detection is advantageous since its average amplitude is less, although this amplitude in the absence of a signal (the case considered) is usually of little importance. The distribution function of the "envelope" of narrow band noise, taken as the distribution function of maxima, is also calculated.

C. E. Shannon (New York, N. Y.).

van der Ziel, A. and Strutt, M. J. O. Suppression of spontaneous fluctuations in $2n$-terminal amplifiers and networks. Physica **9**, 528–538 (1942). [MF 9820]

It is shown that by proper feedback from all but one of the output terminals of a $2n$ ($n>2$) terminal amplifier it is possible, in general, to reduce the signal-to-noise ratio at the output, the noise being assumed to arise in the amplifier. Thus, for example, with n identical amplifiers whose inputs are in parallel, the mean square signal-to-noise ratio of one output can be improved by a factor of n through feedback from the other pairs (this improvement would also occur if the n outputs were connected in series). *C. E. Shannon.*

Borgnis, F. and Ledinegg, E. Der Einfluss einer Phasenfokussierung höherer Ordnung auf die Fourierkomponenten der Strahlstromdichte. Z. Tech. Phys. **23**, 306–312 (1942). [MF 9351]

The original theory of velocity modulation of electron beams as developed by D. L. Webster [J. Appl. Phys. **10**, 501–508 (1939)] is extended to the case of modulation with fundamental and superimposed double frequency term. It is shown that one can obtain in this way focusing of higher order, as well as higher values for the harmonic components of the current density. The theoretical efficiency can be raised from 58% to 74%. *E. Weber* (Brooklyn, N. Y.).

Weissfloch, Albert. Kreisgeometrische Vierpoltheorie und ihre Bedeutung für Messtechnik und Schaltungstheorie des Dezimeter- und Zentimeterwellengebietes. Hochfrequenztech. Elektroak. **61**, 100–123 (1943). [MF 9218]

The parameters of the equivalent passive four-pole are determined by three measurements and graphical means are discussed to represent the generalized characteristics. The most convenient basis is the general linear function of a complex variable and its transformation properties.

E. Weber (Brooklyn, N. Y.).

Dube, G. P. Electrical energy of two cylindrical charged particles. Indian J. Phys. **17**, 189–192 (1943). [MF 9978]

The author considers a cylindrically shaped particle in an electrolytic solution and calculates the electrical potential V in a plane perpendicular to the axis of the particle and going through its center. He reaches this by solving the Debye-Hückel equation V^{-1} div grad V=const., where the boundary condition consists of an assigned value of a derivative of V. Then he considers another particle of the same size as the first one and calculates their mutual potential energy. He shows that the latter has a minimum. All calculations involve Bessel functions.

I. Opatowski (Chicago, Ill.).

Cagniard, Louis. Sur l'extinction des courants d'induction dans une sphère conductrice. Applications géophysiques. C. R. Acad. Sci. Paris **216**, 479–481 (1943). [MF 10026]

The extremely slow extinction of currents induced within a conducting sphere by the sudden suppression of an exciting magnetic field is studied anew with the purpose of obtaining new formulae better adapted to numerical computation. The general solution given in 1883 by H. Lamb is purely theoretical, the series expressing the results being too slowly convergent for practical applications. The new form of solution is perfectly adapted to the numerical applications. Some of the series obtained converge so rapidly that in practice their two first terms are sufficient. As a geophysical application an explanation of the earth's actual magnetic

field is proposed as produced by a sudden suppression, 150 million years ago, of a cosmic uniform magnetic field of 25 oersteds, the nonmagnetic nucleus of the earth having a radius of 3500 klms and a conductivity thirty seven times greater than that of copper. This explanation presupposes the instantaneous propagation of light ($c=\infty$) and disregards the earth's diurnal rotation since the analysis of the problem is done under these two assumptions.

E. Kogbetliantz (Bethlehem, Pa.).

Sotchnev, A. J. A new method for the theoretical investigation of magnetic field of electromagnets. C. R. (Doklady) Acad. Sci. URSS (N.S.) **33**, 24–27 (1941). [MF 9620]

To describe the magnetic field of multi-polar electromagnets, the author restates the differential equations in cylindrical coordinates and gives several special approximate expressions for the field distribution without derivation.

E. Weber (Brooklyn, N. Y.).

Kohler, Max. Die elektrischen und thermischen Eigenschaften von Metallen im Magnetfeld. Ann. Physik (5) **42**, 142–164 (1942). [MF 9239]

The author develops the general theory of electric, thermal and thermo-electric phenomena of crystalline media in a magnetic field. It is found that the tensors of the thermoforce, of the electric and of the thermal conductivity may, in a magnetic field, contain antisymmetric constituents between which simple relations are shown to be valid.

F. W. London (Durham, N. C.).

***von Neumann, Johann. Mathematische Grundlagen der Quantenmechanik.** Dover Publications, N. Y., 1943. 266 pp. $3.50

Photoprint of the (only) German edition of 1932 [J. Springer, Berlin]. A German-English glossary of about 360 scientific and technical terms has been added to the present edition.

Stueckelberg, E. C. G. An unambiguous method of avoiding divergence difficulties in quantum theory. Nature **153**, 143–144 (1944). [MF 10060]

Dobbs, H. A. C. Difficulties in Dirac's representation theory. Philos. Mag. (7) **34**, 651–674, 723–744 (1943). [MF 9558]

The author examines critically Dirac's devices for handling simultaneously the cases of discrete and continuous eigenvalues in quantum mechanics, and in particular the device of "improper" functions such as the δ function. The difficulties he emphasizes are already sufficiently obvious to mathematicians, and he fails to note that many of them may be avoided by the use of Stieltjes integrals.

O. Frink (State College, Pa.).

Roubaud-Valette, Jean. La transformation de Lorentz et la mécanique ondulatoire. C. R. Acad. Sci. Paris **213**, 563–566 (1941). [MF 9177]

Following a procedure of Schrödinger, the author obtains an operator equation from the Lorentz transformation by replacing W/c and P_x by $(h/2\pi i)\partial/\partial T$ and $(-h/2\pi i)\partial/\partial x$, respectively, where W is the energy, P_x a momentum and $T=ct$. This operator equation is applied to a wave function of the form $a\exp(2\pi i\phi/h)$, and equations involving the amplitude a are derived. The same procedure, but with the conjugate replacement of x and T by $(-h/2\pi i)\partial/\partial P$ and $(h/2\pi i)\partial/\partial(W/c)$, respectively, leads from the Lorentz transformation to equations involving the amplitudes A corresponding to a wave function of the form $\sum A(P, W/c)\times\exp(2\pi i\phi/h)$. The corresponding equations in Dirac form for wave functions of four components are also found.

The purpose of all this is to find formulas for the probability that a particle will have coordinates x' and T' in a coordinate system S' when its coordinates x and T in a system S are known, but its energy and momentum are unknown, or, in the second case, when its energy and momentum are known, but its coordinates in S are unknown. (The system S is in uniform motion relative to S'.)

O. Frink (State College, Pa.).

Schrödinger, Erwin. The general unitary theory of the physical fields. Proc. Roy. Irish Acad. Sect. A. **49**, 43–58 (1943). [MF 8993]

A four-dimensional affine space with scalar density, the Lagrangian function a function of the contracted curvature tensor alone, is postulated. The field equations are obtained from a variational principle using this Lagrangian. The gravitational potentials are defined in terms of the symmetric part of the derivatives of the Lagrangian with respect to the components of the contracted curvature tensor and the electromagnetic field strengths are defined in terms of the antisymmetric part of these derivatives. A single vector is interpreted both as the current vector and the vector potential. Hence it is difficult to see how the gauge invariance of the usual theory can be taken into account in the present work. A special Lagrangian is chosen which leads to the Born-Infeld theory of the electromagnetic field. The paper closes with some remarks concerning the introduction of an additional affine connection to take into account the meson fields.

A. H. Taub.

Beck, Guido. Theory of static fields. I. A phenomenological attempt to determine the proper field of an electron. Phys. Rev. (2) **64**, 366–375 (1943). [MF 9709]

Taking the Dirac wave equation as the fundamental relation, the author attempts to calculate the proper field of an electron. By this procedure he hopes that certain difficulties which arise in the classical theories will be avoided. Neither the precise objective of the work nor the results are stated in a readily understandable form. It appears, however, that the results of the attempt are not regarded as entirely satisfactory, and further papers are promised.

L. A. MacColl (New York, N. Y.).

Iskraut, Richard. Bemerkungen zum Energie-Impuls-Tensor der Feldtheorien der Materie. Z. Phys. **119**, 659–676 (1942). [MF 9347]

Following Belinfante's procedure [Physica **6**, 887–898 (1939); these Rev. **1**, 280], a symmetric energy-momentum tensor is defined for a general field theory derived from a Lagrangian L by a variation principle. The decomposition of the angular momentum into orbital momentum and spin momentum is derived using the symmetrical energy-momentum density. The possibility of an additional condition for the Lagrangian or the symmetrical energy-momentum tensor is stressed, which grants the absence of the derivatives of the $g_{\mu\nu}$ in L in the case of the general theory of relativity. It is shown that the additional condition is satisfied in the case of the Dirac theory of spin $\frac{1}{2}$ particles, the Proca theory of spin 1 particles and the Maxwell theory.

W. Pauli (Princeton, N. J.).

de Boer, J. The non spherical potential field between two hydrogen molecules. Physica 9, 363–382 (1942). [MF 9816]

The author calculates the first order perturbation of the interaction of two hydrogen molecules using as basis the so-called Wang wave-function for the unperturbed hydrogen molecule:

$$\psi_{a,b}(i,j)=(z^3/\pi)(e^{-z(r_{ai}+r_{bj})}+e^{-z(r_{aj}+r_{bi})})$$

with i, j the index of the electrons, a, b the index of the protons, $z=1.166$ and r_{ai} the distance between a and i in atomic units. For large distances the interaction reduces to the Coulombian interaction of two quadrupoles of the momenta $\Theta_1=1.282$ and $\Theta_2=\Theta_1=1.534$ in atomic units. To this first order interaction the long range Van der Waals attraction of the second order perturbation is added. For two main orientations of the molecules (parallel to the line joining the two molecule centers and perpendicular to this line and to each other) the energy is calculated numerically as a function of the molecular distance and compared with the spherical force field by which the experimental virial coefficient has been interpreted hitherto. The experimental force field is found to fall within the theoretical limits given by the two main orientations of the molecules.

F. London (Durham, N. C.).

Eljaševič, M. Determination of the form of the oscillations of polyatomic molecules from the symmetry properties. Akad. Nauk. SSSR. Zhurnal Eksper. Teoret. Fiz. 13, 65–84 (1943). (Russian) [MF 9588]

Meister, Arnold G., Cleveland, Forrest F. and Murray, M. J. Interpretation of the spectra of polyatomic molecules by use of group theory. Amer. J. Phys. 11, 239–247 (1943). [MF 9546]

Casimir, H. B. G. and Karreman, G. On the magnetic octupole moment of a nucleus. Physica 9, 494–502 (1942). [MF 9819]

The observations of Tolansky [Proc. Roy. Soc. London. Ser. A. 170, 214–222 (1939)] that the separation of the hyperfine structure of the spectrum of iodine can only be represented by a formula involving the cube of the Landé cosine factor between the nuclear spin and the electronic angular momentum and the interpretation of such terms as arising from the interaction of the electron and the magnetic octupole moment of the nucleus are investigated in detail. General formulae are developed for the magnetic interaction between the nucleus and the electron, and the calculations are carried out for the magnetic octupole interaction of the iodine atom. Numerical results are dependent on only roughly known constants such as the effective nuclear charge and the effective quantum number and the distribution of magnetization inside the nucleus, but it is shown that even with the most extreme assumptions the theoretical values are about a hundred times smaller than the values necessary to explain the observed separations. Hence it is concluded that the interpretation of the observed result must be sought in another direction.

S. Kusaka (Northampton, Mass.).

Kramers, H. A., Belinfante, F. J. und Lubanski, J. K. Über freie Teilchen mit nichtverschwindender Masse und beliebiger Spinquantenzahl. Physica 8, 597–627 (1941). [MF 9539]

The wave equation (1) $(N\kappa+\Gamma^\lambda\partial/\partial x^\lambda)\Psi=0$ is investigated, where Ψ is an undor of rank N and Γ^λ is a vector operator with four components. This is a generalization of the Dirac equation for free electron and is intended to describe particles of different masses and arbitrary spin. For $N>2$, different values of the constant which determines the mass of the particle are possible. This constant can take only one value, however, for the cases $N=1$ (the Dirac electron) and $N=2$ (which leads to the meson theory of Møller and Rosenfeld and of Belinfante). Because of the generality of equation (1), the study of its solutions allows a comparison between, and a unified treatment of, various recent theories of particles of arbitrary spin.

It is shown that the general solution of (1) with preassigned momentum vector is obtained by the superposition of plane wave solutions similar to the plane wave solutions of the Dirac equation. The variational principle which corresponds to equation (1) is shown to lead to explicit expressions for the vector of current and charge density, and for the energy-momentum tensor. Unfortunately, when $N>2$ the expression for the energy obtained in this way is not positive definite. This difficulty cannot be overcome, as in the case $N=1$, by a "hole theory" like that of Dirac, but it is avoided in the theory of Fierz and Pauli by the restriction to particles of lowest mass.

O. Frink.

Heitler, W. and Peng, H. W. On the production of mesons by proton-proton collisions. Proc. Roy. Soc. Irish Acad. Sect. A. 49, 101–133 (1943). [MF 9715]

This paper deals with the calculation of the cross sections for the production of mesons by proton-proton and proton-neutron collisions. The effect of radiation damping is taken into account by the method recently proposed by these authors [Proc. Cambridge Philos. Soc. 38, 296–312 (1942)] and, since in this theory the direct calculation leads to great mathematical complications, the computation is carried out using the method of virtual quanta due to Williams and von Weizäker. The form of the meson theory assumed is that of Møller and Rosenfeld in which the field is a mixture of pseudoscalar and vector mesons with equal masses and coupling constants, and the calculations are made both for the symmetrical and the charged cases.

The method of the virtual quanta is to replace the meson field of the proton moving with velocity nearly equal to c approximately by a superposition of plane waves describing free mesons, and part I of the paper is devoted to the calculation of this meson spectrum. The meson field of the proton is obtained by Lorentz transformation of the static field, and then the spectrum is obtained by Fourier expansion of this field. In part II the cross sections for the scattering of mesons by a nucleon are calculated, and they are carried out for two cases, the nonrelativistic case where the meson energy is small compared to the nucleon rest, and the extreme relativistic case where the meson energy is large compared to the nucleon rest mass. The results of the first two parts are combined in part III to obtain the cross section for the production of mesons by collision of two nucleons. The results show that these cross sections are very large, and that most of the mesons so produced have only a small energy. The detailed comparison of the predictions of this theory with cosmic ray data are made in another paper by these authors and Hamilton [Phys. Rev. (2) 64, 78–94 (1943)].

S. Kusaka.

Ma, S. T. Calculations of the scattering of mesons by the matrix method. Acad. Sinica Science Record 1, 123–129 (1942). [MF 8847]

The scattering of mesons is treated with the help of the matrix formulation of the meson theory as developed by

Kemmer [Proc. Roy. Soc. London. Ser. A. 173, 91–116 (1939); these Rev. 1, 95]. It is shown that this method permits the computation of scattering for initial states with definite polarization, and not only for an average over initial polarization. Applications are given for scattering by an electrostatic field and it is mentioned that the method can be used also for nuclear fields. *L. W. Nordheim.*

Mariani, Jean. Sur les relations qui existent entre le spin et les statistiques. C. R. Acad. Sci. Paris 213, 775–777 (1941). [MF 9656]

This note attempts to give a simple explanation of the relation between the spin and statistics of elementary particles. The consideration is based on the group of rotations about the z-axis and the wave function of a system of particles in the configuration space written as the product of one-particle wave functions. The author does not seem to be acquainted with the work of Pauli [Phys. Rev. (2) 58, 716–722 (1940)] in which this problem is solved.

S. Kusaka (Northampton, Mass.).

Statistical Mechanics

Khintchine, A. Les fonctions convexes et les théorèmes d'évolution de la mécanique statistique. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 7, 111–122 (1943). (Russian. French summary) [MF 9867]

En remplaçant dans l'énoncé des théorèmes d'évolution de la mécanique statistique la fonction $x \log x$ par une fonction convexe et continue arbitraire, on parvient à la fois à une généralisation raisonnable de leur portée et à une simplification essentielle de leurs démonstrations. La méthode est appliquée au théorème classique de Gibbs, aux inégalités de Klein et Pauli et à deux théorèmes de J. von Neumann.

Author's summary.

Khintchine, A. Sur les méthodes analytiques de la mécanique statistique. C. R. (Doklady) Acad. Sci. URSS (N.S.) 33, 438–441 (1941). [MF 9631]

Classical statistical mechanics deals with systems in which the total energy E (that is, Hamiltonian) can be taken to have zero as its least value. From the point of view of phase-space averages, the statistics of the system is determined by the structure function $\Omega(x)$ which can be defined indifferently as the derivative $V'(x)$ of the volume $V(x)$ of that part of phase-space for which $E \leqq x$, or by the equation $\Omega(x) = \int_{E=x} d\Sigma/\text{grad } E$. A second restrictive property of these systems is that for each $\alpha > 0$ the integral $\Phi(\alpha) = \int_0^\infty \Omega(x) e^{-\alpha x} dx$ is finite.

One is concerned with those systems formed by the composition of a large number n of independent systems of this type (to be characterized individually by subscripts), and one desires an asymptotic expression for the resulting total structure function $\Omega(x)$, which is given by the well-known law

$$\Omega(x) = \int \prod_{k=1}^{n} \Omega_k(x_k) dS^{n-1}$$

taken over the $n-1$ dimensional hyperplane in $x_1 \cdots x_n$-space having the equation $x_1 + \cdots + x_n = x$. Inasmuch as this is precisely the formula of composition of independent probabilities for the sum of n variates, one would wish to apply the Laplace-Liapounoff theorem to obtain the asymptotic expression for $\Omega(x)$. But this is impossible, since $\int_0^\infty \Omega_k(x) dx = \infty$; that is, $\Omega_k(x)$ cannot be taken as a probability.

The whole purpose of this paper is to circumvent this difficulty by the simple device of introducing the conjugate distributions

$$u_k^{(\alpha)}(x) = \frac{\Omega_k(x) e^{-\alpha x}}{\Phi_k(\alpha)}, \quad u^{(\alpha)}(x) = \frac{\Omega(x) e^{-\alpha x}}{\Phi(\alpha)},$$

the integrals of which between 0 and ∞ are unity. These functions satisfy the same equation of composition as before:

$$u^{(\alpha)}(x) = \int \prod_{k=1}^{n} u_k^{(\alpha)}(x) dS^{n-1},$$

and, furthermore, $\Phi(\alpha) = \prod_{k=1}^{n} \Phi_k(\alpha)$. On applying the Laplace-Liapounoff theorem to the conjugate distributions, and then returning to the structure functions, one obtains

$$\Omega(x) = \Phi(\alpha) e^{\alpha x} \{(1/(\sigma\sqrt{2\pi})) e^{-(x-m)^2/(2\sigma^2)} + O\},$$

where $O = O((\log^2 n)/n^{\frac{1}{2}})$ for $|x-m| < 2 \log^2 n$, $O = O(1/n)$ for arbitrary x. One finds that $m = -(d/d\alpha) \log \Phi(\alpha)$, $\sigma^2 = (d^2/d\alpha^2) \log \Phi(\alpha)$. For physical reasons, the hitherto undetermined constant α is taken as the root $\alpha = \theta$ of the equation $\epsilon + (d/d\alpha) \log \Phi(\theta) = 0$ (ϵ now being the constant energy), and then θ is identified with $1/kT$ (k = Boltzmann constant, T = absolute temperature).

Thus the foundations of statistical mechanics are reduced in a simple and rigorous manner to classical probability.

B. O. Koopman (New York, N. Y.).

Khintchine, A. Valeurs moyennes des fonctions sommatoires dans la mécanique statistique. C. R. (Doklady) Acad. Sci. URSS (N.S.) 33, 442–444 (1941). [MF 9632]

This note employs the methods and terminology of the one reviewed above, of which it is a continuation.

In a system composed of a large number of small systems ("molecules," in general dissimilar), a function $f(P)$ of a point P in its phase space (P = Hamiltonian variable) is additive (sommatoire) if it is of the form $f(P) = \sum_{i=1}^{n} f_i(P_i)$, P_i being a point in the phase space of the ith molecule. The present note deals with the evaluation of the means $\bar{f} = \sum_{i=1}^{n} \bar{f}_i$, where evidently

$$\bar{f}_i = \int f_i(P_i) (\Omega^{(i)}(E - E_i)/\Omega(E)) dP_i;$$

here $\Omega^{(i)}$ is the structure function of the residual system left from the given one when the ith molecule is omitted, and the integration is throughout the phase space of the ith molecule. By the formulas of the previous note, the integrand can be replaced by an asymptotic approximation, from which one obtains

$$\bar{f}_i = \int f_i(P_i) (e^{-\theta E_i}/\Phi_i(\theta)) dP_i + O((\log^4 n)/n).$$

When, in particular, $f_i(P_i)$ depends only on the energy E_i of the ith system: $f_i(P_i) = \chi_i(E_i)$, then

$$\bar{f}_i = \int_0^\infty \chi_i(x) (e^{-\theta x}/\Phi_i(\theta)) \Omega_i(x) dx = \int_0^\infty \chi(x) u_i^{(\theta)}(x) dx.$$

When E^* and Φ^* are the energy and structure function of a large (many molecular) subsystem of the given one, one obtains $\bar{E}^* = -(d/d\theta) \log \Phi^*$. *B. O. Koopman.*

Khintchine, A. Sur la corrélation intermoléculaire. C. R. (Doklady) Acad. Sci. URSS (N.S.) **33**, 482–484 (1941). [MF 9634]

This note, which is a continuation of the two reviewed above, has for its object the derivation of an asymptotic approximation to the dispersion of additive functions of a system composed of a large number of molecules. The result that such dispersions are small justifies the use of means of such functions instead of the functions themselves in many physical applications.

Indices 1 and 2 refer to two chosen molecules in the system, 12 to their combination, and any two functions $\varphi(P_1)$ and $\psi(P_2)$ of their respective Hamiltonian variables are envisaged. In view of the relation $\sum_{i=1}^{n} E_i = E$, a knowledge of the value of the total energy E of the system induces a statistical correlation between $\varphi(P_1)$ and $\psi(P_2)$ which turns out to be nonnegligible even when n is large. In finding the desired dispersion, it is necessary to evaluate not only the dispersions $D\varphi$ and $D\psi$, but also their correlation coefficient $R(\varphi, \psi)$. The expressions for these quantities are written in terms of the conjugate distributions [the u's of the second preceding review] and, when the technique of that paper is used and the precise form of the Laplace-Liapounoff theorem [see the following review] is introduced, the following is obtained:

$$R(\varphi, \psi) = -((b_1 b_2)^{\frac{1}{2}}/B) R(\varphi, E_1) R(\psi, E_2) + O(n^{-\frac{1}{2}} \log^2 n),$$

$b_1 = DE_1$, $b_2 = DE_2$, $B = D$ (conjugate distribution of system). When $\varphi = \psi = E_1 = E_2$ one finds, in particular,

$$R(E_1, E_2) = -n^{-1} + O(n^{-\frac{1}{2}} \log^2 n).$$

More generally, for any additive function $f(P)$ in the system, $(Df)^{\frac{1}{2}}/f = O(n^{-\frac{1}{2}})$. Other results of a similar nature are derived. *B. O. Koopman* (New York, N. Y.).

Khintchine, A. Sur un cas de corrélation a posteriori. Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 185–195 (1943). (French. Russian summary) [MF 9582]

This paper is devoted to the formulation, study and detailed proof of a theorem which was introduced by the author in a special case in connection with statistical mechanics [cf. the preceding review]. Let $\mathfrak{x}_1, \mathfrak{x}_2, \cdots$ be a sequence of independent variates of probability densities $\omega_1(x_1), \omega_2(x_2), \cdots$, and let $f_1(\mathfrak{x}_1), f_2(\mathfrak{x}_2), \cdots$ be a given set of functions of them; these new variates are likewise independent. They cease to be independent when an experiment gives the information that $\mathfrak{x}_1 + \cdots + \mathfrak{x}_n = a$. The problem is to find an asymptotic expression for the correlation coefficient of any given pair of variates $f_i(\mathfrak{x}_i)$, $f_k(\mathfrak{x}_k)$ $(i, k < n)$ for large n. Under very general but precisely stated restrictions imposed on the ω's and f's, the following formula is derived:

$$R^{(a)}\{f_i(\mathfrak{x}_i), f_k(\mathfrak{x}_k)\} = -((\beta_i \beta_k)^{\frac{1}{2}}/B) R\{\mathfrak{x}_i, f_i(\mathfrak{x}_i)\} \times R\{\mathfrak{x}_k, f_k(\mathfrak{x}_k)\} + O(n^{-1}),$$

where $R^{(a)}$ is the correlation coefficient on the presumption that $\mathfrak{x}_1 + \cdots + \mathfrak{x}_n = a$, while R is that without this supposition; β_i, β_k and B are the dispersions of $\mathfrak{x}_i$, $\mathfrak{x}_k$ and $\mathfrak{x}_1 + \cdots + \mathfrak{x}_n$ (again without assuming the value a).

The proof (lengthy but straightforward) employs the more precise form of the Laplace-Liapounoff theorem involving the first six Hermite polynomials, such as it appears in H. Cramér ["On the composition of elementary errors," Skand. Aktuarietidskr. **1928**, 13–74, p. 71]. Some applications are given. *B. O. Koopman* (New York, N. Y.).

Kronig, R. On a simple statistical property of an ensemble of linear harmonic oscillators. Physica **9**, 113–116 (1942). [MF 9810]

In thermal equilibrium the probabilities of the quantum states are distributed according to an exponential function of the energy. For each state the probability of the coordinate x is the square of a Hermite function of x. Compounding these probabilities and summing out the state, the author finds that the probability of the coordinate irrespective of the state is of normal ("Gaussian") form in x. [He measures the "energy" from the ground state of energy $\frac{1}{2}h\nu$, but this usage does not affect his wave functions.]

A. Blake (Washington, D. C.).

Lichnerowicz, André et Marrot, Raymond. Le H-théorème et la notion de système isolé. C. R. Acad. Sci. Paris **212**, 1074–1077 (1941). [MF 9639]

Starting with Boltzmann's integral equation, it is shown that certain averages obey the equation of continuity or its generalization; these averages are the mean density and components of material flux. By introducing the H-function of Boltzmann regarded as an average, together with certain other related averages which can be regarded as components of a vector-flux of entropy, an inequality of continuity is established. A consideration of these relations, together with some other equations of continuity for the flux of energy, leads to the definition of an "H-isolated" system. No mathematical details are given. *B. O. Koopman.*

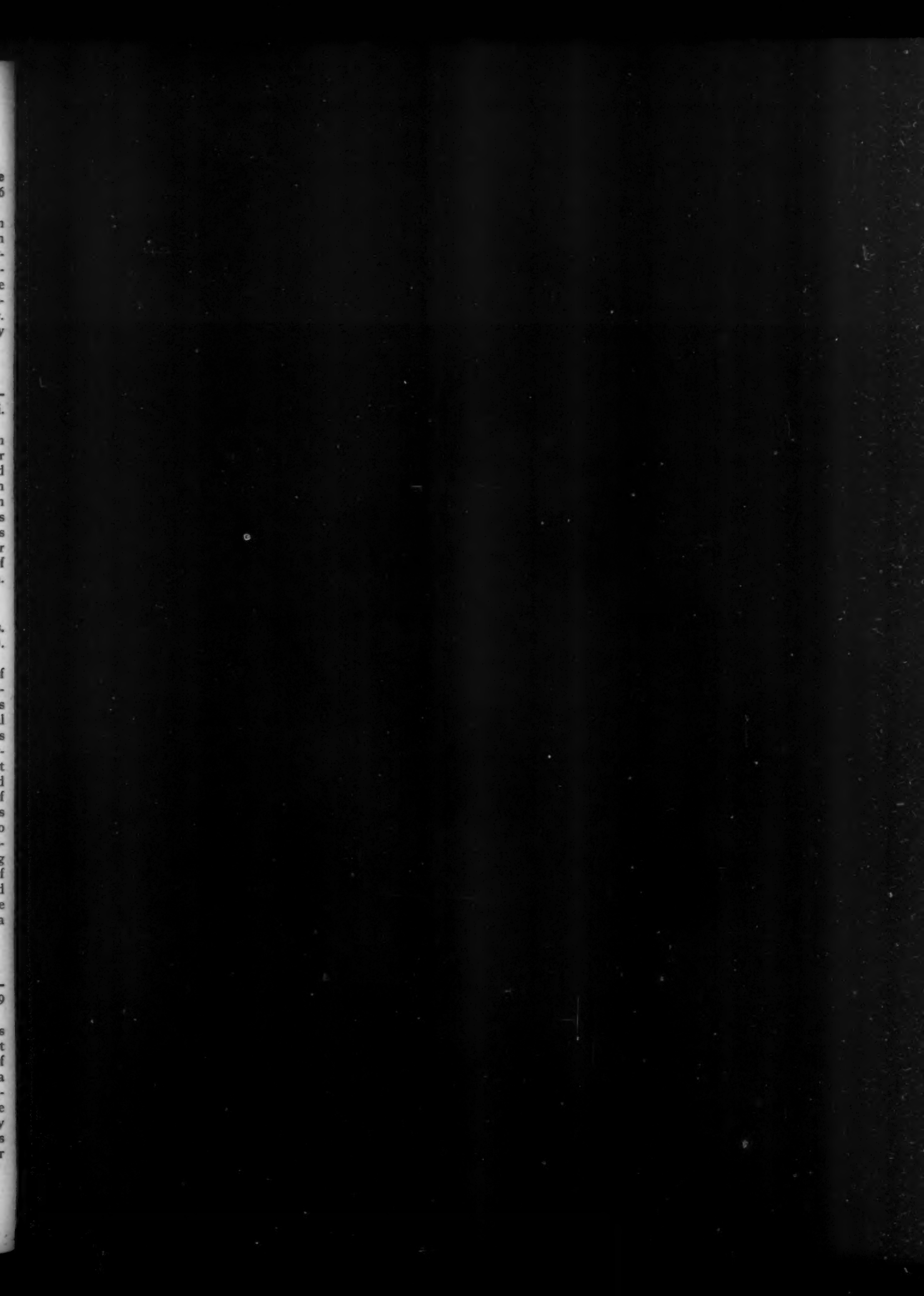
Power, S. The intensity distribution of proper vibrations. Proc. Roy. Irish Acad. Sect. A. **49**, 91–100 (1943). [MF 9716]

For the black body radiation of nondispersive media of variable index of refraction the thermodynamical equilibrium between places of different index of refraction n is established if the density of radiation energy is proportional to the third power of n. In the present paper the analogous problem is being investigated for dispersive media, in particular, for the case of the ψ-waves of wave-mechanics. It is shown that the proportionality with n^3 has to be replaced by a proportionality with $n^2 n_g$, where n_g is "the index of refraction for the group-velocity." In the case of s degrees of freedom the equilibrium density is proportional to $n^{s-1} n_g$. The Boltzmann law follows directly from this distribution and the canonical temperature distribution among the energy states of the system. These statements are, of course, approximate to the extent that they are restricted by the condition that the potential energy (that is, the index of refraction) should not vary appreciably within a range of the order of the wave-lengths in question.

F. London (Durham, N. C.).

Biswas, B. N. Spontaneous energy fluctuation and quantum statistics. Bull. Calcutta Math. Soc. **35**, 55–59 (1943). [MF 9949]

Calculations are made for the spontaneous fluctuations of the following quantities: (a) the energy of a perfect monatomic gas at constant volume under conditions of negligible degeneracy; (b) the number of molecules in a given energy state for the three cases of Maxwell-Boltzmann, Einstein-Bose and Fermi-Dirac statistics; (c) the energy of radiation with a given frequency; (d) the energy of a monatomic solid using the Debye model. The results in general agree with the values obtained earlier by other investigators. *S. Kusaka* (Northampton, Mass.).



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